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Steering Dynamical Systems with Finite Plans and Limited Path Length

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***Abstract*— Complex dynamical systems can be steered by using symbolic input plans. These plans must have a finite descriptive length, and can be expressed by means of words chosen in an alphabet of symbols. In this way, such plans can be sent through a limited capacity channel to a remote system, where they are decoded in suitable control actions. The choice of this symbols is essential to efficiently encode steering plans. To this aim, in this paper, we state the problem of finding symbols maximizing the interval of points reachable by the system along paths with constrained length. We focus on the problem with two symbols, and compare the results with those produced by plans not accounting for the length constraint. Moreover, the behavior of a simple helicopter, steered by both kinds of plans, has been simulated, in order to illustrate the power of the overall control system, and to emphasize the improvements introduced by the new plans.**

I. INTRODUCTION

In this paper we address the problem of generating efficient input plans P amenable of transmission through a finite capacity channel, to steer complex dynamical systems of the type

$$\dot{x} = f(x, u), \tag{1}$$

$x \in X \subseteq \mathbb{R}^n$, $u \in U \subset \mathbb{R}^r$, between neighborhoods of given initial and final equilibria. The plans must have low specification complexity, in terms of the minimum number of bits necessary to represent P, and bounded path length, i.e. the length of each path must be upper bounded.

Motivations to find solutions to this problem come from the following observations. First, the development of steering technique for complex physical plants has generally to tackle with both kinematic and dynamic system constraints. In recent years, several approaches inspired to the kinodynamic paradigm [1], [2], and consisting in trying to solve these problems simultaneously, have been presented. Secondly, dealing with physical systems and complex control frameworks, such as those based on hierarchically abstracted levels of decision or networked control, usually involves additional issues related to limited communication and storage resources.

Several important contributions have addressed this problem by proposing different type of symbolic control schemes, e.g. [3]–[5]. Moreover, [6] has shown that feedback can substantially reduce the specification complexity to reach a certain goal state, i.e. the minimum description length of a motor program solving P.

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In this vein, we showed in [7], [8] that finite plans can be efficiently found for a wide class of systems, by suitable use of feedback. We proposed a symbolic encoding scheme ensuring that a *control language* is obtained whose action on the system has the desirable properties of additive groups, i.e. the actions of control words are invertible and commute. Under this hypothesis, the reachable set obtained by using actions of this language, becomes a lattice. These properties simplify the planning, allowing a solution plan P to be found in polynomial time, provided that the length constraint is not considered.

More precisely, a suitable (dynamic) feedback encoding permits us to transform any flat system to:

$$z^+ = z + \bar{H}\mu, \quad \bar{H} \in \mathbb{R}^{n \times n}, \; \mu \in \mathbb{Z}^n. \tag{2}$$

Once reduced to this special form, we addressed the problem of optimally choosing finite input sets in order to minimize the specification complexity of plans.

It is worth noting that, even for feedback linearized systems, the quantity of information needed to specify such steering plans may not be small. Our method allows to reduce such complexity as it was remarked in [7].

In this paper we extend the previous technique by considering also the length constraint. We state the general problem of finding the optimal symbols, ensuring the maximal interval coverage, without violating an upper bound on the path length. The solution for the case with 2 symbols is provided and the plans generated by the new choice of symbols are compared with the old ones, thus enlightening the improvements obtained.

II. BACKGROUND RESULTS

Symbolic control makes use of elementary control events, or *quanta*, to build complex control actions. A finite or countable set $\mathcal{U}$ of control quanta can be *encoded* by associating its elements with symbols in a finite set $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$. Furthermore, letters from the alphabet Σ can be employed to build words of arbitrary length. Let Σ^* be the set of such strings, including the empty one.

In [7] authors showed the importance of realizing encodings guaranteeing simple composition rules for the action of words in a sublanguage $\Omega \subset \Sigma^*$. In particular, it is advisable that the global action of a command string is independent from the order of application of each control symbols in Ω. It has been proved also, that, under this hypothesis, it is always possible to make the reachable set from any point in X a *lattice*. This fact ensures that, in suitable state and input coordinates, the system (1) takes on the form (2). Moreover, with this choice of Ω, we can ensure that for system (1) the following stronger property holds:

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Definition 1: A control system $\dot{x} = f(x,u)$ is *additively* (or *lattice*) *approachable* if, for every $\varepsilon > 0$, there exist a control quantization $\mathcal{U}_\varepsilon$ and an encoding $E^* : \Omega \mapsto \mathcal{U}_\varepsilon^*$, such that: i) actions of Ω commute and are invertible, and ii) for every $x_0, x_f \in X$, there exists x in the Ω–orbit of x_0 with $\|x - x_f\| < \varepsilon$.

Once the system is in the form (2), a further optimization problem in the input set, can be stated to the aim of minimizing the specification complexity of plans. To this purpose, we can set the tolerance $\varepsilon = 1$ and assume $\bar{H} = I$, thus reducing the system (2) to

$$z^+ = z + u. \tag{3}$$

This system can be treated componentwise, hence it will be sufficient to consider (3) with $z \in \mathbb{R}$.

In [9]–[11] the following optimization problem has been analyzed:

Problem 1: For fixed integers $m > 0$ and $k > 0$, find the best choice of an integer control set $\mathcal{U} = \{0, \pm u_1, \ldots, \pm u_m\}$ such that the reachable set from the origin in at most k steps contains the maximum interval of integers $\mathcal{I}_\mathcal{U}(M) = \{-M, -M+1, \ldots, M\} \subset \mathbb{Z}$.

A complete solution to this problem for $m = 2, 3, 4$ and any k is given in [10] along with a conjectured asymptotic formula for every m. We report here the explicit formulae for the optimal choice of controls for $m = 2, 3$. For $m = 2$ we simply obtain $u_1 = k$ and $u_2 = k + 1$. For $m = 3$ we get:

$$u_3 = \begin{cases} \frac{k^2}{4} + \frac{3}{2}k + \frac{5}{4} & \text{if } k \text{ is odd} \\ \frac{k^2}{4} + \frac{3}{2}k + 1 & \text{if } k \text{ is even,} \end{cases}$$
$$u_2 = u_3 - 1$$
$$u_1 = \begin{cases} u_3 - \frac{k+1}{2} - 1 & \text{if } k \text{ is odd} \\ u_3 - \frac{k}{2} - 2 & \text{if } k \text{ is even.} \end{cases}$$

For $m = 2, 3, 4$ and $k \gg m$, for the largest values in $\mathcal{U}$ it holds asymptotically $u_m \sim \left(\frac{k}{m-1}\right)^{m-1}$. Given $2m + 1$ controls one can thus reach in k steps a region of size

$$M \sim \frac{k^m}{(m-1)^{m-1}}.$$

III. Problem $\max M \ \min \ell$

So far we have considered symbols only maximizing the interval to be covered, but they have the drawback of producing very long paths. A system steered using these symbols, hence, will have large overshoots in state variables or may oscillate until approaching the final position. In order to avoid these problems, we can add a further constraint on the path length to the original optimization problem. Thus, the new formulation is as follows:

Problem 2: For fixed integers $m > 0$ and $k > 0$, find the best choice of an integer control set $\mathcal{U} = \{0, \pm u_1, \ldots, \pm u_m\}$ such that the reachable set from the origin in at most k steps contains the maximum interval of integers $\mathcal{I}_\mathcal{U}(M) = \{-M, -M+1, \ldots, M\} \subset \mathbb{Z}$ and such that the path length from the origin to $N \in \mathcal{I}_\mathcal{U}(M)$, $\ell_\mathcal{U}(0, N)$, satisfies the following inequality

$$\ell_\mathcal{U}(0, N) \le \alpha\, |N| + \beta, \tag{4}$$

with

$$\ell_\mathcal{U}(0,N) = \inf \left\{ \sum\nolimits_i |\alpha_i u_i| : \sum\nolimits_i \alpha_i u_i = N,\ \alpha_i \in \mathbb{Z}, \sum\nolimits_i |\alpha_i| \le k \right\}.$$

A. Problem with $m = 2$: $u_1 = 1$ is optimal

In order to achieve a deeper insight in the general problem, we consider here the case $m = 2$, thus $\mathcal{U} = \{0, \pm u_1, \pm u_2\}$ and assume $0 < u_1 < u_2$.

Let us first prove that the choice $u_1 = 1$ is optimal, at least for $\alpha \in [1, 2]$ and k larger than β.
Notice that $\alpha - 1$ expresses the error, in percentage, with which $\ell_\mathcal{U}(0, N)$ can exceed $|N|$. Therefore, if we assume that a reasonable error should be between 1% and 20%, then $\alpha \in [1.01, 1.2]$. On the other side, β should be a small positive integer (again represents an allowed error for $\ell_\mathcal{U}(0, N)$, now not in percentage), while k is large for the interesting cases.

To achieve the desired result, let us first assume that $u_1 > 1$. Then to reach the point $N = 1$, we must use also the control u_2. Let α_1, α_2 be such that:

$$1 = \alpha_1 u_1 + \alpha_2 u_2,$$

and $\ell_\mathcal{U}(0, 1) = |\alpha_1 u_1| + |\alpha_2 u_2|$. To verify the length constraint we must have:

$$|\alpha_1 u_1| + |\alpha_2 u_2| = \ell_\mathcal{U}(0, 1) \le \alpha + \beta.$$

Then we get the constraint:

$$|\alpha_2 u_2| \le \alpha + \beta - |\alpha_1 u_1| < \beta.$$

and, in particular:

$$u_2 < \beta.$$

On the other side, the length constraint in case $u_1 = 1$ for $N = 1$ is obviously satisfied. Even more, all points up to $N = k$ are reached with $\ell_\mathcal{U}(0, N) = |N|$. Thus the length constraints can be read as:

$$u_2 \le k.$$

Now the optimal value of u_2 is determined in both cases using also the coverage constraint. However, if $u_1 > 1$ then $u_2 < \beta$ and M will not exceed βk, while if $u_1 = 1$ then u_2 can be of the order of k (in fact we will see that $u_2 \sim k$) and M can well be of order k^2 (we will get $M \sim k^2/2$). In conclusion, the case $u_1 = 1$ will permit to have much larger values of M than $u_1 > 1$.

B. The optimal control in case of large k

Now we want to compute the value of the optimal control u_2, for the case $\alpha \in (1, 2)$ and $u_1 = 1$, assuming that $k >> \beta > 1$.

In this case the maximal M is given by

$$M(u_2, \gamma) = \gamma u_2 + k - \gamma \tag{5}$$

where $0 \leq \gamma \leq k$ and u_2 have to be found.

The problem has two constraints: a *coverage constraint* and a *length constraint*.

The coverage constraint requires that each integer in the interval $\mathcal{I}_{\mathcal{U}}(M)$ is written by means of $u_1 = 1$ and u_2 in, at most, k steps. Assume, for simplicity, $N > 0$. Consider the intervals $[(\gamma' - 1)\, u_2, \gamma' u_2] \cap \mathbb{N}$ with $0 \leq \gamma' \leq k$, and in particular the last one with $\gamma' = \gamma$. The greatest integer in this interval that can be represented with a positive combination ($\alpha_i \geq 0$) of 1 and u_2, is given by $(\gamma - 1)\, u_2 + k - \gamma + 1$. Hence, integers in the interval $[(\gamma - 1)\, u_2 + (k - \gamma + 1) + 1, \gamma u_2 - 1] \cap \mathbb{N}$ can be written only with $\alpha_1 < 0$. The smallest integer in this interval, that can be represented with $\alpha_1 < 0$ is $\gamma u_2 - (k - \gamma)$. In order to ensure the complete coverage of this interval, the following inequality must hold

$$\gamma u_2 - (k - \gamma) \leq (\gamma - 1)\, u_2 + (k - \gamma + 1) + 1.$$

Solving in u_2, yields

$$u_2 \leq u_{2cov} \triangleq 2\,(k - \gamma + 1)\,. \tag{6}$$

It is straightforward to see that if $u_2 \leq u_{2cov}$, then any interval with $\gamma' \leq \gamma$ is completely covered.

As concerns the length constraints, it is worth noting that the path length can be greater than $|N|$ only if the coefficients α_i have not all the same sign. With the assumption of $N > 0$ and $u_1 = 1$, we focus our attention only on the integers in the intervals $\mathcal{I}_{\gamma'} = \left[\underline{N}_{\gamma'}, \overline{N}_{\gamma'}\right] \cap \mathbb{N}$ with $\underline{N}_{\gamma'} = (\gamma' - 1)\, u_2 + (k - \gamma' + 1) + 1$, $\overline{N}_{\gamma'} = \gamma' u_2 - 1$ and $0 \leq \gamma' \leq k$. It is easy to verify that $\underline{N}_{\gamma'} = \arg \max\limits_{N \in \mathcal{I}_{\gamma'}} \ell_{\mathcal{U}}(0, N)$. Therefore, if $\underline{N}_{\gamma'}$ violates the length constraint (4), then the whole interval $\mathcal{I}_{\gamma'}$ cannot belong to $\mathcal{I}_{\mathcal{U}}(M)$ as $\underline{N}_{\gamma'}$ is its left bound. Whilst, if $\underline{N}_{\gamma'}$ satisfies the length constraint, the whole interval $\mathcal{I}_{\gamma'}$ can belong to $\mathcal{I}_{\mathcal{U}}(M)$. $\underline{N}_{\gamma'}$ can be written with $\alpha_1 < 0$ as $\underline{N}_{\gamma'} = \gamma' u_2 - \delta$ for some $0 \leq \delta \leq k - \gamma'$. The value of δ can be found by equating the two definitions of $\underline{N}_{\gamma'}$

$$\delta = u_2 - k + \gamma' - 2. \tag{7}$$

Notice that the length constraint (4) relating γ' and u_2 is stricter the smaller is γ'. Indeed we have

$$\ell_{\mathcal{U}}(0, \underline{N}_{\gamma}) = \gamma' u_2 + \delta \leq \alpha\,(\gamma' u_2 - \delta) + \beta$$
$$u_2\,(\gamma'\,(1 - \alpha) + \alpha + 1) \leq (\alpha + 1)\,(k - \gamma' + 2) + \beta.$$

Then, we can simply consider the case $\gamma' = 1$ and rewrite the length constraint as:

$$u_2 \leq \frac{\alpha + 1}{2}(k + 1) + \frac{\beta}{2}. \tag{8}$$

Notice that, as expected above, such constraint essentially impose u_2 to be bounded by k.

On the other side, from (6), we notice that the larger is γ the stricter is the coverage. Finally, the solution to our problem is equivalent to maximize the function $M(u_2, \gamma)$ in (5), over the set determined by the constraints $0 \leq \gamma \leq k$, $0 \leq u_2$, (8) and (6), see Figure 1. By direct computations,

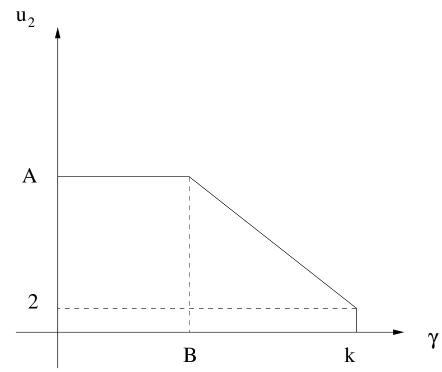


Fig. 1. Maximization set determined by the constraints $0 \leq \gamma \leq k$, $0 \leq u_2$, (8) and (6).

the values A and B in the figure are given by:

$$A = \frac{\alpha + 1}{2}(k + 1) + \frac{\beta}{2}, \qquad B = \frac{3 - \alpha}{4}(k + 1) - \frac{\beta}{4}.$$

Here we used the assumption $k >> \beta$ to make sure that B is positive.

The gradient of our function is:

$$\nabla M = \begin{pmatrix} u_2 - 1 \\ \gamma \end{pmatrix}.$$

The only critical point is $\gamma = 0$ and $u_2 = 1$, which is clearly a minimum for M, thus not interesting for us.

Then we have to compute the maximum on the lines l_1: $u_2 = A$ with $0 \leq \gamma \leq B$, and l_2: $u_2 = 2(k - \gamma + 1)$ with $B \leq \gamma \leq k$. On the line l_1, the function M is clearly increasing w.r.t. γ so the maximum is obtained precisely for $\gamma = B$ and is:

$$M = B \cdot (A - 1) + k$$

For the line l_2, we use the Lagrange multiplier method to find critical points, and thus solve:

$$\left\{ \begin{array}{l} 0 = g(\gamma, u_2) = u_2 - 2(k - \gamma - 1) \\ 0 = \frac{\partial(M + \lambda g)}{\partial \gamma} = (u_2 - 1) + 2\lambda \\ 0 = \frac{\partial(M + \lambda g)}{\partial u_2} = \gamma + \lambda \end{array} \right. ,$$

from which the unique critical point is

$$\gamma = \frac{k}{2} + \frac{1}{4}, \qquad u_2 = k + \frac{3}{2}. \tag{9}$$

Now, to make sure that the point is inside the line l_2, we impose the condition

$$(\alpha - 1)k \geq 2 - \alpha - \beta,$$

which is tantamount to $\beta \geq 1$. Looking at the level curves of M one easily realizes that this is in fact the maximum. However, to make sure, we compute the value of M at this point:

$$M = \frac{k^2}{2} + \frac{9k}{4} + \frac{1}{8}.$$

On the other side the leading term of M at the point (B, A) is:

$$\frac{-\alpha^2 + 2\alpha + 3}{8}(k + 1)^2,$$

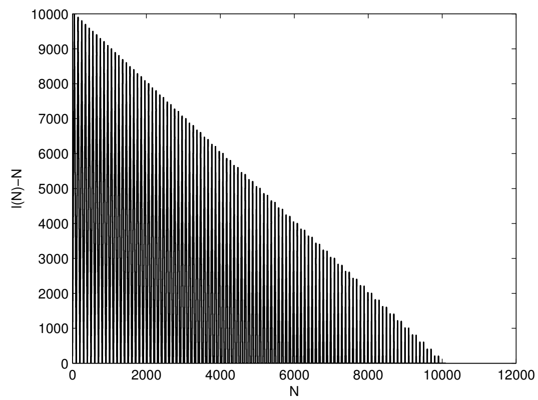


Fig. 2. Overlengths $\ell_{\mathcal{U}}(0,N) - N$ vs N for the old choice of symbols.

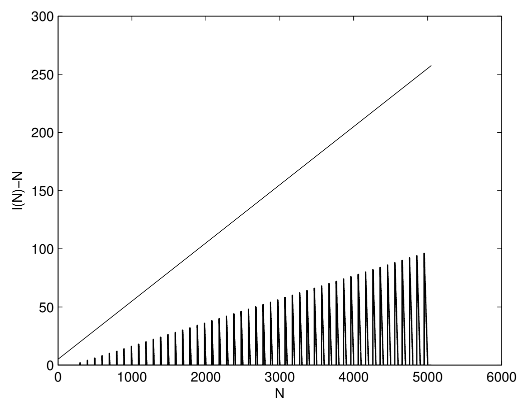


Fig. 3. Overlengths $\ell_{\mathcal{U}}(0,N) - N$ vs N for the new choice of symbols.

and the coefficients of k^2 is strictly less than $1/2$ for $\alpha > 1$. Concluding the maximum of M is obtained at the point (9)

1) Example: Let us compare the improvements in the path length introduced by the new choice of symbols w.r.t. the old one. In Section II it was shown that for $m = 2$ and k steps, the optimal choice of symbols maximizing the interval $\mathcal{I}_{\mathcal{U}}(M)$ is $u_1 = k$ and $u_2 = k+1$. This ensures the coverage of a large interval as $M \sim k^2$, but produces very long paths for small $N \in \mathcal{I}_{\mathcal{U}}(M)$. Indeed, with $k = 100$, hence with $u_2 = 101$ and $u_1 = 100$, we have the overlengths $\ell_{\mathcal{U}}(0,N) - N$ vs N shown in Fig. 2.

Considering also the length constraint and fixing $\alpha = 1.05$, i.e. a percentage error of 5%, and $\beta = 5$, we have, from (9), that the new choice of symbols is $u_1 = 1$ and $u_2 \sim k = 100$. The interval covered with these symbols is shorter: $M \sim \frac{k^2}{2} = 5\ 10^3$ instead of 10^4, but the overlengths are greatly reduced where not zeroed. In Fig. 3 $\ell_{\mathcal{U}}(0,N) - N$ vs N is shown for the new choice of symbols. It is worth noting that the overlength increases for high values of N, but never exceeds the length constraint (the thin line in the figure).Both cases are built choosing for each N the shortest path. For paths having the same length, the one guaranteeing a fewer number of steps, has been considered.

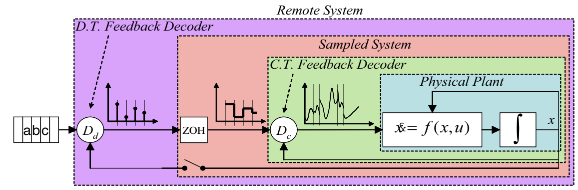


Fig. 4. Depiction of the nested feedback encoding scheme.

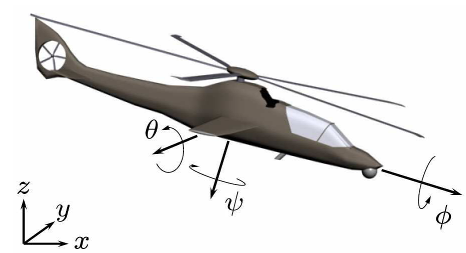


Fig. 5. Depiction of a simple helicopter model.

IV. Simulations

In this section, we illustrate how the proposed choice of input symbols can be used to steer a simple model of helicopter, by means of plans with low specification complexity. Moreover, we show the efficacy of the new choice of symbols in reducing the path length, by comparing old and new plans for the same steering problem.

To begin with, recall from [8] that any nonlinear differentially flat system can be reduced to form (3), by the so-called "nested feedback encoding" scheme (see Fig 4). We also recall that a feedback encoding scheme consists in associating to each input symbol a control value u that depends on the symbol itself, on the current state x of the system, and on its structure [7]. In the specific case, the nested feedback encoding is composed of an inner continuous-time feedback loop, a time-discretization stage, and an outer discrete-time feedback loop. The inner loop realizes a (possibly dynamic) feedback linearization reducing the system to chains of integrators, and the outer loop enables the system to "accept" symbols from the specified control language eventually achieving additive-approachability.

Referring to Fig. 5, a simple model of helicopter can be obtained, by considering it as a rigid body of mass m and inertia J, actuated by the thrust T of the main rotor, and the torque τ of the tail rotor. Let (x,y,z) and (ϕ,θ,ψ) denote position and orientation of a coordinate frame attached to the helicopter center of mass. Then, the helicopter dynamics reads:

$$\begin{cases} M\,\ddot{x} = T\,S_\theta\,, \\ M\,\ddot{y} = -T\,C_\theta\,S_\phi\,, \\ M\,\ddot{z} = T\,C_\theta\,C_\phi - M\,g\,, \\ J\,\ddot{\psi} = \tau\,C_\theta\,C_\phi\,, \end{cases} \tag{10}$$

where g is the gravity acceleration. For the sake of simplicity, we will assume that variable angles ϕ (roll) and θ (pitch) represent two additional inputs, even though they are actually steered by means of aileron and fore-aft cyclic control (this is a standard assumption, see e.g. [12]).

The dynamic model in (10) is transformed into a linear system by the dynamic feedback realized in the inner loop of Fig. 4. This is achieved as follows. Let $y_1 = x$, $y_2 = y$, $y_3 = z$, $y_4 = \psi$ be the system outputs. Differentiating twice w.r.t. time, yields

$$\begin{cases} \ddot{y}_1 = \frac{T\,S_\theta}{M}\,, \\ \ddot{y}_2 = -\frac{T\,C_\theta\,S_\phi}{M}\,, \\ \ddot{y}_3 = \frac{T\,C_\theta\,C_\phi}{M} - g\,, \\ \ddot{y}_4 = \frac{\tau\,C_\theta\,C_\phi}{J}\,. \end{cases} \tag{11}$$

In (11), input controls are nonlinearly coupled, and hence it is not easy to linearize the system by means of static feedback. Moreover, adding one integrator before each input channel, and differentiating (11) once more, yields:

$$\begin{bmatrix} x^{(3)} \\ y^{(3)} \\ z^{(3)} \\ \psi^{(3)} \end{bmatrix} = \begin{bmatrix} \frac{S_\theta}{M} & \frac{T\,C_\theta}{M} & 0 & 0 \\ -\frac{C_\theta\,S_\phi}{M} & \frac{T\,S_\theta\,S_\phi}{M} & -\frac{T\,C_\theta\,C_\phi}{M} & 0 \\ \frac{C_\theta\,C_\phi}{M} & -\frac{T S_\theta C_\phi}{M} & -\frac{T\,C_\theta\,S_\phi}{M} & 0 \\ 0 & -\frac{\tau\,S_\theta\,C_\phi}{J} & -\frac{\tau\,C_\theta\,S_\phi}{J} & \frac{C_\theta\,C_\phi}{J} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} =$$

$$= D(T, \theta, \phi, \tau)\, u\,,$$

where the system's state has been extended by defining the following auxiliary variables: $\dot{T} = u_1, \dot{\theta} = u_2, \dot{\phi} = u_3$, and $\dot{\tau} = u_4$.

Under the hypothesis that $T \neq 0$, $\theta \neq \pm\frac{\pi}{2}$, $\phi \neq \pm\frac{\pi}{2}$, and $\theta \neq -\arcsin\left(\frac{C_\phi^2}{S_\phi}\right)$, matrix D is nonsingular, and the system can be exactly linearized by the choice $u = D^{-1}v'$. With respect to the new input v', the system's dynamics is indeed composed of four chains of integrators, i.e. $y_1^{(3)} = v_1'$, $y_2^{(3)} = v_2'$, $y_3^{(3)} = v_3'$, $y_4^{(3)} = v_4'$.

Consider again system (1), and the associated equilibrium equation $f(x, u) = 0$. Let the equilibrium set be $\mathcal{E} = \{x \in X | \exists u \in U, f(x, u) = 0\}$. We want the equilibrium set $\mathcal{E}$ to be a lattice, but as it is known from [11], it is impossible to steer the state of a system among points in $\mathcal{E}$ while remaining in $\mathcal{E}$, except for special cases. This motivates the search of policies for *periodic* steering of systems among equilibria.

The outer feedback loop of Fig. 4 indeed allows to achieve such a property on $\mathcal{E}$. First, suitable change of coordinates to the state and input spaces are applied in order to transform the system into Brunovsky canonical form. It is worth noting that in this form, the system is decomposed into r subsystems each of dimension κ_i, where κ_i denote the Kronecker control-invariant indices, and the equilibrium set $\mathcal{E}$ has a simple structure. Indeed, letting $\mathbf{1}_{\kappa_i} \in \mathbb{R}^{\kappa_i}$ denotes a vector with all components equal to 1, we have that for each κ_i-dimensional subsystem, the equilibrium states are $\bar{\xi}_i = \alpha_i \mathbf{1}_{\kappa_i}$, $\alpha_i \in \mathbb{R}$, hence

$$\mathcal{E} = \left\{\bar{\xi} | \bar{\xi} = \mathrm{diag}\left(\alpha_1 I_{\kappa_1}, \cdots, \alpha_r I_{\kappa_r}\right) \mathbf{1}_n\right\}$$

The lattice mesh size in Brunovsky coordinates is given by $\gamma_i = \frac{2\varepsilon}{\|\zeta_i\|}$, where

$$\begin{bmatrix} \zeta_1 & \zeta_2 \cdots & \zeta_r \end{bmatrix} = S\, \mathrm{diag}\left(\mathbf{1}_{\kappa_1}, \cdots, \mathbf{1}_{\kappa_r}\right),$$

and S is the Brunovksy state coordinate change. Finally we recall that input symbols actually applied to the system in the Brunovsky coordinate, are scaled by factors γ_i, for $i = 1, \ldots, r$. Hence the input sets are given by $\mathcal{W}_i = \gamma_i\, \mathcal{U}_i$.

Observe that the equilibrium set of the helicopter is

$$\mathcal{E} = \{x \ \in \ \mathbb{R}^{12} | x = (\alpha_x, 0, 0, \alpha_y, 0, 0, \alpha_z, 0, 0, \alpha_\psi, 0, 0)\}\,.$$

Now, apply the discrete-time feedback encoding of Fig. 4 with unit sampling time $t = 1s$, and compute the Brunovsky change of coordinates. In the new coordinates, the equilibrium manifold is given by

$$\mathcal{E} = \{(\beta_x \mathbf{1}_3, \beta_y \mathbf{1}_3, \beta_z \mathbf{1}_3, \beta_\psi \mathbf{1}_3)\}\,,$$

where $\alpha_x = \zeta_1\, \beta_x$, $\alpha_y = \zeta_2\, \beta_y$, $\alpha_z = \zeta_3\, \beta_z$, $\alpha_\psi = \zeta_4\, \beta_\psi$, and $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta = 1$.

When building a lattice for the system, it is reasonable to ask for a tolerance ε_1 on the x, y, and z coordinates which are measured in meters, and a different one, ε_2, on the ψ variable which is instead measured in radiants. Take e.g. as numerical values $\varepsilon_1 = 1$m and $\varepsilon_2 = 0.01$rad, hence it holds $\gamma_1 = \gamma_2 = \gamma_3 = 2$ and $\gamma_4 = 0.02$. Furthermore, we choose $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}_3 = \mathcal{U}_4$.

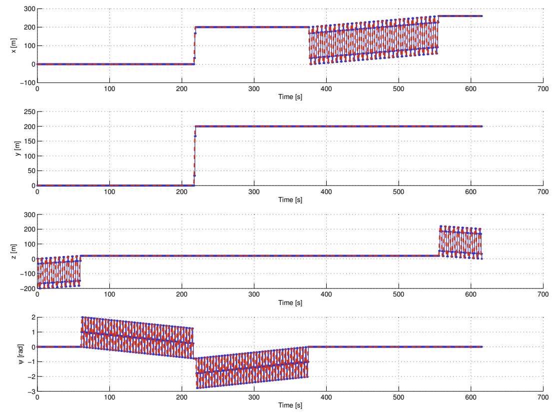
As in section III-B.1, we choose $m = 2$ and $k = 100$, hence we have that $\mathcal{U}_{old} = \{0, \pm 100, \pm 101\}$ is the optimal set satisfying only the coverage constraint, whereas $\mathcal{U}_{new} = \{0, \pm 1, \pm 100\}$ is the set of symbols satisfying also the length constraint. Furthermore, the following task has been specified for the helicopter's motion: lift up of a relative altitude of 20m from the actual position, rotate of an angle $\psi = \pi/4$ rad while hovering, travel horizontally of a relative displacement $(200\text{m}, 200\text{m})$, rotate of an angle $\psi = -pi/4$ rad, travel horizontally of 60m along the x-axis, and finally go down to the initial altitude.

As shown in Fig. 2, the use of the symbol set $\mathcal{U}_{old}$ yields very long paths. This fact may cause the system's state to overshoot or to oscillate while approaching the final position, depending on how the symbols are ordered within the plan (recall that symbols can be arbitrarily ordered due to the commutative property of actions on Ω). Fig. 6-a shows the oscillatory system's behavior, and Fig. 6-b reports the corresponding plan.

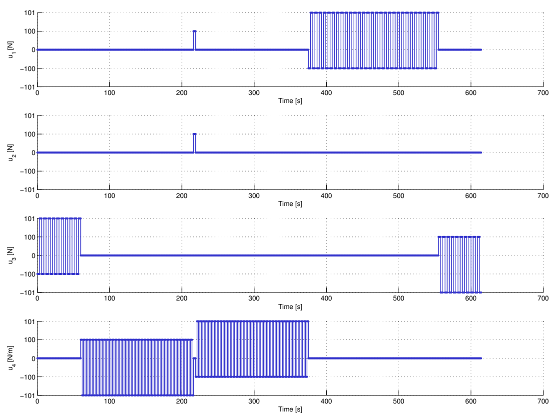
On the contrary, the use of the symbol set $\mathcal{U}_{new}$ effectively reduces the path length, and avoids the oscillations (see Fig. 7-a). Fig. 7-b reports the corresponding input plan.

V. Conclusions

This paper described methods for steering complex dynamical systems by signals with finite-length descriptions. Systems tractable by symbolic control under encoding include all controllable linear systems, nilpotent driftless nonlinear systems, and (dynamically) feedback-linearizable systems. We focused mainly on the search of symbols used to encode steering plans, ensuring the maximal interval coverage and producing paths of bounded length. We analyzed the problem with two symbols, and tested the results on a simple model of helicopter. The improvements introduced by the new symbols in the motion of the helicopter, motivates future efforts in the solution of the general problem with arbitrary number of symbols. Moreover, the new symbols,

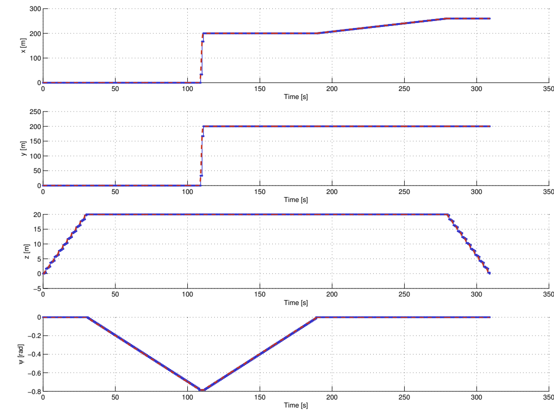


a)

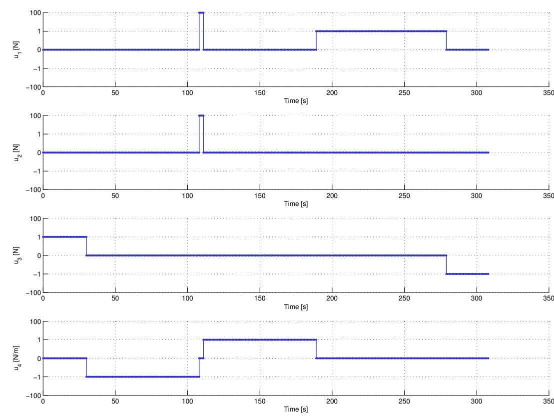


b)

Fig. 6. Evolution of the helicopter's state performing the specified task (a), and corresponding symbolic inputs from the set $\mathcal{U}_{old}$ (b).



a)



b)

Fig. 7. Evolution of the helicopter's state performing the specified task (a), and corresponding symbolic inputs from the set $\mathcal{U}_{new}$ (b).

preventing system to have large overshoot or to oscillate, can provide a valuable aid in solving the problem of planning with obstacles.

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References

[1] S. M. LaValle and P. Konkimalla, "Algorithms for computing numerical optimal feedback motion strategies," *Int. Jour. of Robotics Research*, vol. 20, no. 9, pp. 729–752, 2001.
[2] S. M. LaValle, *Planning Algorithms*. Cambridge University Press, 2006.
[3] V. Manikonda, P. S. Krishnaprasad, and J. Hendler, "A motion description language and hybrid architecture for motion planning with nonholonomic robots," in *Proc. Int. Conf. on Robotics and Automation*, 1995.
[4] M. Egerstedt, "Motion description languages for multimodal control in robotics," in *Control Problems in Robotics*, ser. STAR, A. Bicchi, H. Christensen, and D. Prattichizzo, Eds. Springer-Verlag, 2003, no. 4, pp. 75–89.
[5] E. Frazzoli, M. A. Dahleh, and E. Feron, "Maneuver-based motion planning for nonlinear systems with symmetries," *IEEE Trans. on Robotics*, vol. 21, no. 6, pp. 1077–1091, 2005.
[6] M. Egerstedt and R. W. Brockett, "Feedback can reduce the specification complexity of motor programs," *IEEE Trans. on Automatic Control*, vol. 48, no. 2, pp. 213–223, 2003.
[7] A. Bicchi, A. Marigo, and B. Piccoli, "Feedback encoding for efficient symbolic control of dynamical systems," *IEEE Trans. on Automatic Control*, vol. 51, no. 6, pp. 987–1002, 2006.
[8] A. Fagiolini, L. Greco, A. Bicchi, B. Piccoli, and A. Marigo, "Symbolic control for underactuated differentially flat systems," in *Proc. IEEE Int. Conf. on Robotics and Automation*, 2006, pp. 1649–1654.
[9] Y. Chitour, A. Marigo, and B. Piccoli, "Time optimal control for quantized input systems," in *Proc. IFAC Workshop on Nonlinear Control Systems (NOLCOS'04)*, 2004.
[10] A. Marigo, "Optimal input sets for time minimality in quantized control systems," *Math. Control Signals Systems*, vol. 18, no. 2, pp. 101–146, May 2006. [Online]. Available: http://dx.doi.org/10.1007/s00498-005-0156-5
[11] A. Bicchi, A. Marigo, and B. Piccoli, "Improving efficiency of finite plans by optimal choice of input sets." in *HSCC*, ser. Lecture Notes in Computer Science, J. P. Hespanha and A. Tiwari, Eds., vol. 3927. Springer, 2006, pp. 108–122.
[12] M. J. van Nieuwstadt, "Trajectory generation for nonlinear control systems," Ph.D. dissertation, Cal. Inst. of Tech., 1997.