Feedback Encoding for Efficient Symbolic Control of Dynamical Systems

Antonio Bicchi, *Fellow, IEEE,* Alessia Marigo, and Benedetto Piccoli,

***Abstract*— The problem of efficiently steering dynamical systems by generating input plans is considered. Plans are considered which consist of finite–length words constructed on an alphabet of input symbols, which could be e.g. transmitted through a limited capacity channel to a remote system, where they can be decoded in suitable control actions. Efficiency is considered in terms of the computational complexity of plans, and in terms of their description length (in number of bits).**

We show that, by suitable choice of the control encoding, finite plans can be efficiently built for a wide class of dynamical systems, computing arbitrarily close approximations of a desired equilibrium in polynomial time. The paper also investigates how the efficiency of planning is affected by the choice of inputs, and provides some results as to optimal performance in terms of accuracy and range.

***Index Terms*— Symbolic control, Specification complexity, Hybrid Logic-Dynamical Systems.**

I. INTRODUCTION

IN this paper we consider the problem of planning inputs to steer a controllable dynamical system of the type

$$\dot{x} = f(x,u),\ \ x \in X \subseteq \mathbb{R}^n, u \in U \subset \mathbb{R}^r \qquad (1)$$

between neighborhoods of given initial and final states. As a solution, we seek a *finite plan*, i.e. an input function which admits a finite description. We are interested in plans with short description length (measured in bits) and low computational complexity. Particular attention is given to plans among equilibrium states, regarded as nominal functional conditions.

Motivations to study this problem come from a growing number of applications requiring to steer physical plants, consisting of dynamical systems capable of complex behaviours, by hierarchically abstracted levels of decision, planning and supervision, i.e. by logic control. In the control literature, methods for generation of reference trajectories have been often considered as feedforward components in a two degree of freedom controller design ([1]). In this spirit, several authors have addressed the problem of reducing the trajectory generation problem for complex systems by planning for simpler, lower dimensional ones, by e.g. kinematic reductions [2], group symmetries [3], [4], flatness-theory tools [1], or hierarchical system abstractions [5].

With respect to that framework, an additional concern about the complexity of describing plans is introduced whenever communication or storage limitations are in place. Particularly fitting to this perspective are examples from robotics, where input symbols may represent commands (aka *behaviors*, or *modes*.) For instance, for autonomous mobile rovers, high level plans may be comprised of sequences of motion primitives such as `wander, look_for, avoid_wall,` etc.; in the control of humanoids (see e.g. [6]), symbols are encountered such as `walk, run, stop, squat,` etc.. An operational specification for such systems is naturally given in terms of the language built on symbols. The capability of such languages to encode the richest variety of tasks by words of the shortest length, is a crucial aspect when dealing with realistic conditions. Consider for instance the case where the robotic agent receives its reference plans from a remote high-level control center through a finite capacity communication channel, or plans are exchanged in a networked system of a large number of simple semi-autonomous agents. In general, it can be assumed that robots are capable of accepting finitely-described reference signals, and can implement a finite number of possible different feedback strategies via the use of embedded controllers, according to the received messages.

Several important contributions have appeared in recent years addressing different instances of such symbolic control problem, e.g. [4], [7], [8]. A general framework for such systems and problems can be traced back to ideas on Motion Description Languages in [9]. The line of research addressing finite hyerarchic abstractions of continous systems via bisimulations ([5], [10], [11]) has several contact points with the one presented in this paper, although the type of methods and results are thus far quite distinct. Of direct relevance to work presented here is the quantitative analysis of the specification complexity of input sequences for a class of automata, presented in [12]. The key result there is that feedback can substantially reduce the specification complexity (i.e., the description length of the shortest admissible plan) to reach a certain goal state.

In this paper, we address similar questions as in [12] for continuous dynamical systems. The problem is tougher than for automata, in that continuous systems are not finitely describable themselves, and exact plans have in general the cardinality of the continuum. Considering straightforward quantizations of the inputs, on the other hand, does not help much, as the reachable set may result in a mere collection of scattered

Manuscript received November 11, 2004; revised September 5, 2005. This work was supported by the European Commission through the Network of Excellence contract IST-2004-511368 "HYCON - HYbrid CONtrol: Taming Heterogeneity and Complexity of Networked Embedded Systems", and Integrated Project contract IST-2004-004536 "RUNES - Reconfigurable Ubiquitous Networked Embedded Systems".

Antonio Bicchi is with the Interdepartmntal Research Center "Enrico Piaggio", University of Pisa, Italy.

Alessia Marigo is with the Department of Mathematics, Università di Roma - La Sapienza, Rome, Italy.

Benedetto Piccoli is with the C.N.R. Istituto per le Applicazioni del Calcolo "E. Picone", Rome, Italy.

points ([13]–[15]), thus making planning computations very complex. The main contribution of this paper is to show that, again by suitable use of feedback, finite plans can be efficiently found for a wide class of systems.

More precisely, we show that by introducing *control encoding* of a symbolic input language, we can compute in polynomial time plans whose specification complexity is logarithmic in the size of the region to be covered. In our context, we postulate that control decoders are available and embedded on the remotely controlled plant. Decoders receive symbols from the planner, and translate them in suitable control actions, possibly based on locally available state information.

Whenever the proposed method of symbolic encoding applies, a *control language* is obtained whose action on the system has the desirable properties of additive groups, i.e. the actions of control words are invertible and commute. Furthermore, under the action of words in this language, the reachable set becomes a lattice. Finite–length plans to achieve arbitrarily close approximations of a given equilibrium can thus be computed in polynomial time. Under this point of view, the contribution of this paper can be regarded as an extension of planning techniques in [15] (only applicable so far to driftless nonlinear systems in so-called "chained-form"), to a much wider class of systems, most notably systems with drift. This objective is achieved by three main novel ideas which are developed in this paper: i) the introduction of feedback encoding, which affords the wealthy of feedback-equivalence results in the nonlinear systems literature; ii) the study on the minimal specification complexity for interval-filling controls, derived from concurrent work of number-theoretic nature, and iii) the concept and technique of periodic steering for systems with drift.

By virtue of feedback encoding, complex nonlinear systems — indeed, the same class of differentially flat systems [16] considered in [1] — can be abstracted (at least locally) to a linear system. Planning for flat systems can then be achieved in a linear setting, hence projected back on the original systems by feedback decoding. This process is thoroughly illustrated in the paper by an example of a MIMO nonlinear model of an underactuated mechanical system.

A. Problem Description

Consider again the control system in (1). We assume that for inputs u in a rich enough class of functions (e.g., the space of bounded functions L^∞), the system (1) is completely controllable, i.e. for any given two points x_0, x_f, a *plan* (i.e., a finite-support input function $u : [t, T + t]$) exists that steers (1) from x_0 to x_f.

Such plan would generically require an infinite–length description. Because we are interested in finitely describable plans, i.e. concatenations of only a finite number of elementary control actions, only approximate steering can be achieved in general. We therefore study the following question:

> **Problem** Π: Given a compact subset $\mathcal{M} \subseteq X$ and a tolerance ε, provide a specification P of plans such that, for any pair $(x_0, x_f) \in \mathcal{M}^2$, a plan in P exists such that system (1) is steered from x_0 to within an ε-neighborhood of x_f.

We consider *efficient* a solution to this problem such that plans in P have i) low computational complexity, in terms of the number of elementary computations to be executed to find P, and ii) low specification complexity, in terms of the minimum number of bits necessary to represent P (cf. [12]).

B. A simple example

To appreciate the difference between possible solutions to problem Π, consider a discrete-time linear controllable system (the $+$ superscript denoting the forward shift operator)

$$x^+ = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in U,$$

with $U = \{u \in \mathbb{R}^r : \|u\| \le r_U\}$ and $\mathcal{M}$ the hypercube of half-size M in the r-dimensional equilibrium subspace $\mathcal{E}$ of X.

A direct approach might be the following: introduce a finite point set $\Lambda = \{x^i\} \subset \mathcal{M}$ of dispersion $\rho = \max_{x \in \mathcal{M}} \min_i \|x - x_i\|$. For N sufficiently large, and every pair (x^i, x^j), determine a control sequence u^{ij} of length N such that $x^j - A^N x^i - R_N u^{ij} = 0$, where $R_N = [A^{N-1}B| \cdots |B]$. Notice that, to cover $\mathcal{M}$ in the worst-case direction, it is necessary that $\underline{\sigma}(R_N) r_U \sim M$, where $\underline{\sigma}(R_N)$ denotes the minimum singular value of R_N. This forces a lower bound on the time horizon N. Using $\beta \sim \log_2(2r_U/\mu)$ bits to represent a real number, we get an approximate control sequence $\hat{u}^{ij}$ such that $\|\hat{u}^{ij} - u^{ij}\| \le \mu$. The desired tolerance on planning is achieved if $2\rho + \mu\overline{\sigma}(R_N) \le \varepsilon$, with $\overline{\sigma}(R_N)$ the maximum singular value of R_N. Fixing e.g. $\mu\overline{\sigma}(R_N) \sim \rho \sim \varepsilon/3$, for large M/ε and N we get that the asymptotic specification complexity of P is given by

$$\mathcal{C}(P) \sim \alpha r N \log_2\left(\frac{M}{\varepsilon} c(R_N)\right), \tag{2}$$

with $\alpha = \left(\frac{2M}{\varepsilon}\right)^r$ and $c(R_N) = \overline{\sigma}(R_N)/\underline{\sigma}(R_N) \ge 1$ the condition number of R_N.

As a result of Theorem 15, the approach introduced in this paper leads instead to a specification complexity for the same problem of the order of

$$\mathcal{C}(P) \sim \alpha r \log_2\left(\frac{M}{\varepsilon}\right). \tag{3}$$

Plans provided by our method are developed over a time horizon N' which is minimal for a given r_U, hence $N \ge N'$. Setting $N = N'$ it is then possible to compare the specification complexities of the two methods, as illustrated in fig. 1.

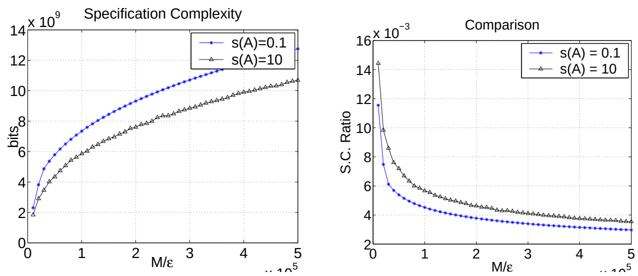
II. Symbolic Control and Encoding

Symbolic control is inherently related to the definition of elementary control events, or atoms, or *quanta*:

Definition 1: A *control quantum* is a couple (u, T) where $u : X \to L^\infty(\mathbb{R}^+ \times X, U)$ and $T : X \to \mathbb{R}^+$. The set of control quanta is denoted by $\widetilde{\mathcal{U}}$.

A control quantum (u, T) is naturally associated with a map $\phi_{(u,T)} : X \to X$, such that, given $x_0 \in X$ and denoting $u_{x_0} = u(x_0)$, $\phi_{(u,T)}(x_0)$ is the solution at time $T(x_0)$ of the Cauchy problem

$$\begin{cases} \dot{x} = f(x, u_{x_0}(t, x)) \\ x(0) = x_0. \end{cases} \tag{4}$$



a) b)

Fig. 1. a) Specification complexity in (2), as a function of M/ε in the range of 10^5. Each data point is obtained as an average over 50 systems ($n = 30$, $r = 1$), generated randomly by the Matlab DRSS function and scaled to have spectral radius as in the legend. b) Ratios between specification complexities (3) and (2), for data generated as above.

Definition 2: A *control quantization* consists in assigning a finite or countable set $\mathcal{U} \subset \widetilde{\mathcal{U}}$. A *(symbolic) control encoding* on a control quantization is a map $E : \Sigma \to \mathcal{U}$, where $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ is a finite set of symbols.

Given a control quantization and an encoder, we have the diagram

$$\Sigma \xrightarrow{E} \mathcal{U} \xrightarrow{\phi} \mathcal{D}(X),$$

where $\mathcal{D}(X)$ denotes the group of automorphisms on X. This can be extended in an obvious way to

$$\Sigma^* \xrightarrow{E^*} \mathcal{U}^* \xrightarrow{\phi^*} \mathcal{D}(X),$$

where Σ^* is the set of words formed with letters from the alphabet Σ, including the empty string ϵ. We assume $\phi \circ E(\epsilon) = Id(X)$, i.e. the identity map in $\mathcal{D}(X)$. An action of the monoid Σ^* on X is thus defined. Given a quantization $\mathcal{U}$ and an initial point x_0, $R(\mathcal{U}, x_0)$ denotes the reachable set from x_0 under $\mathcal{U}$, i.e. the Σ^*–orbit for the action defined via $\phi^* \circ E^*$.

In general, being the action of Σ^* just a monoid, the analysis of its action on the state space can be quite hard, and the structure of the reachable set under generic quantized controls can be very intricated (even for linear systems: see e.g. [13]–[15]). However, suitable choices of encoding of symbolic control may simplify the analysis.

A. Additive Group Actions and Lattices

We focus our attention on designing encodings that achieve simple composition rules for the action of words in a sublanguage $\Omega \subset \Sigma^*$, such that

$$\forall \omega \in \Omega, \exists h(\omega) \in \mathbb{R}^n : \forall x \in X, (\phi^* \circ E^*(\omega))(x) = x + h(\omega), \tag{5}$$

and

$$\forall \omega_1 \in \Omega, \exists \bar{\omega}_1 \in \Omega : (\phi^* \circ E^*(\omega_1)) \circ (\phi^* \circ E^*(\bar{\omega}_1)) = Id(X). \tag{6}$$

The additivity rule (5) implies that actions commute, i.e. $\forall \omega_1, \omega_2 \in \Omega, (\phi^* \circ E^*(\omega_1)) \circ (\phi^* \circ E^*(\omega_2)) = (\phi^* \circ E^*(\omega_2)) \circ (\phi^* \circ E^*(\omega_1))$: therefore, the global action is independent from the order of application of control words in Ω. Rules (5) and (6) amount to requiring that the sublanguage Ω acts on the states as an additive group. As a consequence, the reachable set from any point in X under the concatenation of words in Ω is a set Λ generated by vectors $h(\omega), \omega \in \Omega$,

$$\Lambda = \{h(\omega_1)\lambda_1 + \cdots + h(\omega_N)\lambda_N + |\lambda_i, \in \mathbb{Z}, N \in \mathbb{N}\}.$$

When Λ can be generated by linearly independent vectors, it is called a *lattice*. This happens for instance when $h(\omega) \in \mathbb{Q}^n$, $\forall \omega \in \Omega$, or else when all words in Ω consist of concatenations of only n words in Σ^* which produce independent vectors $h(\omega)$. Under rules (5), (6), a choice for Ω always exists such that Λ is a lattice. In such hypotheses, Ω acts on $\mathbb{R}^n$ as $\mathbb{Z}^n$, hence, in suitable state and input coordinates, the system takes on the form

$$z^+ = z + \bar{H}\mu, \quad \bar{H} \in \mathbb{R}^{n \times n}, \; \mu \in \mathbb{Z}^n. \tag{7}$$

A further important concern is that system (1) under symbolic control, maintains the possibility of approximating arbitrarily well all reachable equilibria in its state space, for suitable choices of symbols.

Definition 3: A control system $\dot{x} = f(x, u)$ is *additively* (or *lattice*) *approachable* if, for every $\varepsilon > 0$, there exist a control quantization $\mathcal{U}_\varepsilon$ and an encoding $E^* : \Omega \mapsto \mathcal{U}_\varepsilon^*$ with $card(\mathcal{U}_\varepsilon) = q \in \mathbb{N}$, such that: i) the action of Ω obeys (5), (6), and ii) for every $x_0, x_f \in X$, there exists x in the Ω–orbit of x_0 with $\|x - x_f\| < \varepsilon$.

Remark 1: The reachable set being a lattice under quantization does not imply additive approachability. For instance, consider the example used in [17] to illustrate the so–called kinodynamic planning method ([18]–[20]). This consists of a double integrator $\ddot{q} = u$ with piecewise constant encoding $\mathcal{U} = \{u_0 = 0, u_1 = 1, u_2 = -1\}$ on intervals of fixed length $T = 1$. The sampled system reads

$$\begin{bmatrix} q \\ \dot{q} \end{bmatrix}^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} u,$$

hence

$$\begin{array}{rcl} q(N) & = & q(0) + N\dot{q}(0) + \sum_{i=1}^{N} \frac{2(N-i)+1}{2} u(i) \\ \dot{q}(N) & = & \dot{q}(0) + \sum_{i=1}^{N} u(i). \end{array}$$

The reachable set from $q(0) = \dot{q}(0) = 0$ is

$$R(\mathcal{U}, 0) = \left\{ \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \lambda, \, \lambda \in \mathbb{Z}^2 \right\}.$$

The quantization thus induces a lattice structure on the reachable set. The lattice mesh can be reduced to any desired ε resolution by scaling U or T. However, the actions of control quanta do not compose according to rule (5): indeed, $\phi^*(u_1 u_2) \neq \phi^*(u_2 u_1)$ (for instance, $\phi^*(u_1 u_2)(0, 0) = (1, 0)$, while $\phi^*(u_2 u_1)(0, 0) = (-1, 0)$).

The following theorem motivates the interest in seeking control encodings for additive approachability.

Theorem 1: For an additively approachable system, a specification P for problem Π can be given in polynomial time.

Proof: Consider a feedback encoding ensuring additive approachability. Arrange a sufficient number q of action vectors $h(\omega_i), \omega_i \in \Omega$ in the columns of a matrix $H \in \mathbb{R}^{n \times q}$. The reachable set from x_0 is thus a lattice given by $R(\Omega, x_0) =$

$x_0+\Lambda$, where $\Lambda=\{H\lambda|\lambda\in\mathbb{Z}^q\}$. Additive approachability guarantees that the dispersion of Λ can be bounded by $\frac{1}{2}\varepsilon$, hence, $\forall x_f$, $\exists y\in\Lambda:\|x_f-x_0-y\|\le\varepsilon$. Finding a plan to x_f is thus reduced to solving the system of diophantine equations

$$y = H\lambda. \tag{8}$$

Each lattice coordinate λ_i represent directly the number of times the control word ω_i, hence the corresponding sequence of control quanta, is to be used to reach the goal. Due to additivity of the action, the order of application of the ω_i is ininfluent. The linear integer programming problem (8) can be solved in polynomial time with respect to the state space dimension n and p. Indeed, write H in Hermite normal form, $H=[L\ 0]\ U$, where $L\in\mathbb{R}^{n\times n}$ is a nonnegative, lower triangular, nonsingular matrix, and $U\in\mathbb{Q}^{m\times m}$ is unimodular (i.e., obtained from the identity matrix through elementary column operations). Once the Hermite normal form of H has been computed (which can be done off-line in polynomial time [21], [22]), all possible plans to reach any desired configuration y are easily obtained as

$$\lambda = U^{-1}\begin{bmatrix} L^{-1}y \\ \mu \end{bmatrix},\ \forall\mu\in\mathbb{Z}^{m-n}.$$

■

B. Reducing the specification complexity

We now address the specification complexity for problem Π for a system in form (7). Without loss of generality to the purposes of this section, we can set the tolerance $\varepsilon=1$ and assume $\bar{H}=Id$, thus reducing to system

$$z^+ = z + u. \tag{9}$$

This system can be treated componentwise, hence it will be sufficient to consider (9) with $z\in\mathbb{R}$.

We address the steering problem Π taking the following point of view: fixed the cardinality of the control set and the time horizon, choose control values maximizing the size of the region to be filled with reachable points. More precisely, we formulate the following problem.

Problem 1: For fixed integers $m>0$ and $N>0$, find the best choice of an integer control set $\mathcal{W}=\{0,\pm v_1,\ldots,\pm v_m\}$ such that the reachable set from the origin in N steps contains the maximum interval of integers $I(M)=[-M,-M+1,\ldots,M]\subset\mathbb{Z}$.

Clearly, the cardinality $2m+1$ of the control set and the number N of steps determine the specification complexity, while M describes the size of the region which can be reached. Thus maximizing M is the same as maximizing the reachable region for a fixed specification complexity.

Problem 1 is a number theoretical problem, related but not equivalent to the well-known “Frobenius postage stamp proble”. More precisely, the postage problem seeks to maximize the minimum postage fee not realizable using stamps from a finite set of m possible denominations. For the classical postage problem, only results for small values of m are known, see [23]. The main difference with Problem 1 is the positivity of stamp denominations, while control values from $\mathcal{W}$ are also negative. Although this difference has substantial technical implications, the difficulty of the two problems is comparable.

N	1	2	3	4	5	6	7
v_1	1	3	5	8	11	15	19
v_2	2	4	7	10	14	18	23
v_3	3	5	8	11	15	19	24
M	3	10	24	44	75	114	168

N	1	2	3	4	5	6	7
v_1	1	3	7	13	19	29	41
v_2	2	6	9	18	27	36	52
v_3	3	7	11	20	29	39	55
v_4	4	8	12	21	30	40	56
M	4	16	36	84	150	240	392

TABLE I

OPTIMAL INTERVAL-FILLING INPUT VALUES FOR SYSTEM (9) FOR $m=3$ (ABOVE) AND $m=4$ (BELOW).

Problem 1 was solved for $m=2,3,4$ and any N in [24]. We report here the explicit formulae for the optimal choice of controls for $m=2,3$. For $m=2$ we simply obtain $v_1=N$ and $v_2=N+1$. For $m=3$ we get:

$$v_3=\begin{cases} N^2/4+3/2\,N+5/4 & \text{if } N \text{ is odd}\\ N^2/4+3/2\,N+1 & \text{if } N \text{ is even,}\end{cases}$$

$$v_2 = v_3 - 1,$$

$$v_1=\begin{cases} v_3-\frac{N+1}{2}-1 & \text{if } N \text{ is odd}\\ v_3-\frac{N}{2}-2 & \text{if } N \text{ is even.}\end{cases}$$

Table I reports the maximum interval of the horizontal line which can be covered with unit resolution and different word lengths N, along with the actual values of the different control sets, for $m=3$ and $m=4$. All values in table I, except N, should be scaled by the desired resolution ϵ.

For $m=2,3,4$ and $N>>m$, for the largest value in $\mathcal{W}$ it holds asymptotically

$$v_m \sim (\left\lfloor\frac{N}{m-1}\right\rfloor+2)^{(m-1)}. \tag{10}$$

Given $2m+1$ controls one can thus reach in N steps a region of size

$$M\sim N^m/m^m. \tag{11}$$

In [24], it is conjectured that (10), (11) hold for every m.

Let us now go back to Problem Π for system (9), and compute the specification complexity for optimized control values. To describe plans covering the region of size M, a sequence of length N of symbols from an alphabet of size $2m+1$ should be given. This would result on a specification complexity of $N\lceil\log_2(2m+1)\rceil$.

A further reduction of specification complexity can be obtained by using run–length encoding (RLE) for control symbols. RLE consists in replacing repeated runs of a single symbol in an input stream by a single instance of the symbol and a run count. This compression method is particularly well suited for our method, because of the commutativity of symbols in control strings.

The following proposition holds for $m \leq 4$, and is a consequence of the conjecture in [24] for larger control sets:

Proposition 2: The specification complexity of Problem Π for a system in the form (9) is given asymptotically by $\mathcal{C} \sim \log_2(M/\varepsilon)$.

Proof: Consider the solution $\mathcal{W}$ to Problem 1. By commutativity of the action group, we can assign, for each possible control value $v_i \in \mathcal{W}$, an integer of size at most N, specifying how many times the control v_i must be used. In this way, the control sequence requires $(2m+1)\lceil\log_2(N)\rceil$ bits, or rather, by exploiting the symmetry of the symbol set and using sign-magnitude representation, $(m+1)\,(1+\lceil\log_2(N+1)\rceil)$ bits. From (11), we get $m \sim \log_2(M)/\log_2(N)$. We conclude reinserting the previously normalized tolerance ε. ■

We finally remark that the solution of Problem 1 at the same time minimizes the number of steps N for given region size, specification complexity, and tolerance. In particular, from (11) we get

Proposition 3: For a fixed tolerance ε and specification complexity, with optimally chosen controls the number N of steps necessary to cover a region of size M is $o(M)$. In other terms, the size M of the reachable set increases faster than the word length N.

III. Feedback Encoding

A few examples of possible control encoding schemes of increasing generality are pictorially described in fig. 2.

In *piecewise constant encoding*, each input symbol in Σ is associated with a control quantum $q_i = (u_i, T_i)$ whereby u_i is constant for fixed time T_i (fig. 2-a). Input quantization, as defined in most part of the literature, is an instance of this scheme. The action of piecewise constant inputs on general systems is typically unstructured [13]. However, the particular class of chained-form driftless systems was shown in [15] to be additively approachable by rational piecewise-constant control quanta.

Piecewise smooth encoding, where T_i is fixed, and u_i are smooth functions of time not depending on the state (fig. 2-b), may allow for more powerful planning. For instance, different u_i's may represent pieces of extremal controls to be pasted together in an approximate optimal control scheme (cf. e.g. [25]). Development of reference trajectories on a functional basis, as in e.g. [1], [26] can also be regarded as an instance of this scheme.

Feedback encoding consists in associating to each symbol a control input u that depends on the symbol itself, on the current state of the system, and on its structure. The scheme can be regarded as generated by defining a feedback $u = f(x, r)$ embedded on system (4), and a piecewise constant encoding on the reference r, and can be realized either directly in continuous time (fig. 2-c), or indirectly through sampling (fig. 2-d). If the encoding incorporates memory elements, e.g. additional states ξ are used to define $u = f(x, \xi, r)$ with $\dot{\xi} = \alpha(\xi, x, r)$, the feedback encoding is referred to as dynamic.

A. Planning by Feedback Encoding

The method of feedback encoding avails symbolic control with powerful results from the literature on feedback equivalence of dynamical systems. In this section we show how this can be exploited to apply the planning method of theorem 1 to rather general classes of systems.

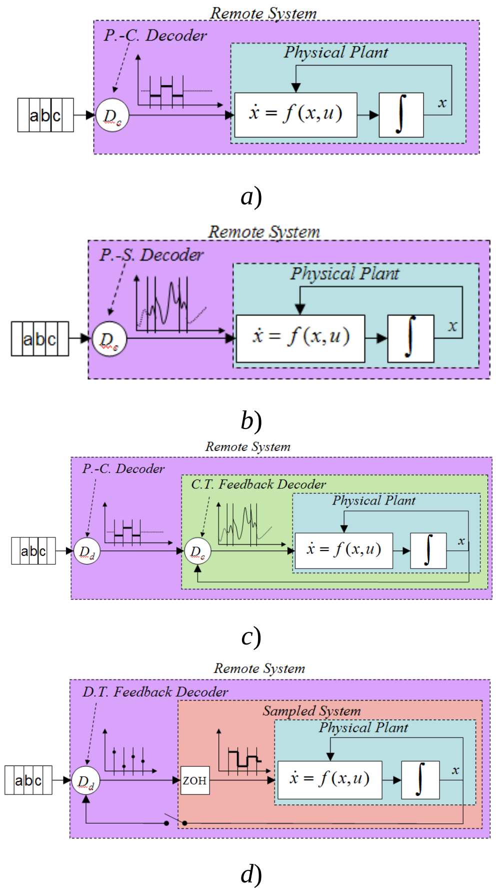


Fig. 2. Four examples of symbolic encoding of control. Symbols transmitted through the finite-capacity channel are represented by letters in the leftmost blocks. From the top: a) piecewise constant encoding; b) piecewise smooth encoding; c) continuous-time feedback encoding; d) discrete-time feedback encoding.

A first consequence that follows almost for free from concepts introduced above concerns the kinematic model of a car with n trailers ([27], [28]). Indeed, by results of [29], we know that the n-trailer system is locally feedback equivalent to chained form, hence additively approachable by feedback encoding. By theorem 1, finite plans to arbitrary accuracy can be found in polynomial time.

A similar result holds indeed for a much broader class of nonlinear systems.

Theorem 4: Linear-in-control, driftless, controllable nonlinear systems

$$\dot{x} = \sum_{i=1}^{p} g_i(x) u_i, \quad x \in \mathbb{R}^n$$

whose control Lie algebra is nilpotent, are locally additively approachable by feedback encoding.

Proof: We start noting that, by defining feedback encoding according to the local feedback equivalence result of [30], we can reduce to the study of strictly triangular systems of

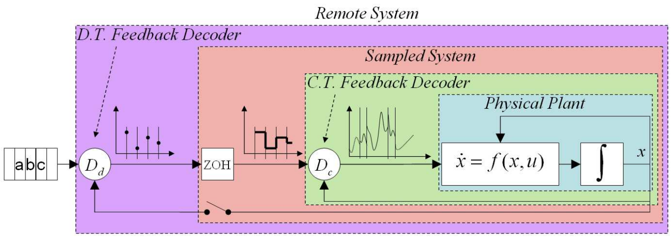


Fig. 3. Nested discrete-time continuous-time feedback encoding.

the form

$$\begin{array}{rcl} \dot{x}_1 & = & \sum_{i=1}^p g_1^i(x_2,\ldots,x_p)u_i \\ \dot{x}_2 & = & \sum_{i=1}^p g_2^i(x_3,\ldots,x_p)u_i \\ & \vdots & \\ \dot{x}_{p-1} & = & \sum_{i=1}^p g_{p-1}^i(x_p)u_i \\ \dot{x}_p & = & \sum_{i=1}^p g_p^i u_i \end{array} \tag{12}$$

with $x = [x_1, x_2, \ldots, x_p] \in \mathbb{R}^{n_1+n_2+\cdots+n_p} = \mathbb{R}^n$, $u = [u_1, ..., u_p] \in \mathbb{R}^{n_p}$, $n_p = p$, and the coefficients $g_j^i(\cdot)$ are polynomials. A lemma proving that systems in the strictly triangular form (12) are additively approachable is given in the appendix. ■

Among controllable systems without drift, i.e. systems for which any state is equilibrium with zero control, the problem of finite planning by symbolic control remains open for non-nilpotent systems.

We now turn our attention to systems with drift, i.e. systems which possess an autonomous dynamics independent of applied inputs. More precisely, consider again system (1)

$$\dot{x} = f(x,u), \quad x \in X \subseteq \mathbb{R}^n,\, u \in U \subset \mathbb{R}^r$$

and the associate equilibrium equation $f(x,u) = 0$. Let the equilibrium set be $\mathcal{E} = \{x \in X | \exists u \in U, f(x,u) = 0\}$. We say that system (1) has drift if $\mathcal{E}$ has lower dimension than X.

In the rest of this paper we will deal with the planning problem for systems with drift, and in particular with generating trajectories to join different equilibrium configurations. This focus is consistent with usual practice in control, where equilibrium configurations typically correspond to nominal working conditions for a system (possibly up to group symmetries, see e.g. [4]).

Among systems with drift, linear systems are the simplest, yet their analysis encompasses the key features and difficulties of planning. Indeed, our strategy to attack the general case consists of reducing to planning for linear systems via feedback encoding. To achieve this, we introduce a further generalized encoder (still encompassed by the above definition of control quanta), i.e. the *nested feedback encoding* described in fig. 3. In this case, an inner continuous (possibly dynamic) feedback loop and an outer discrete-time loop – both embedded on the remote system – are used to achieve richer encoding of transmitted symbols.

Additive approachability for linear systems, by discrete-time feedback encoding (see fig. 3), is proved in theorem 9 below. By using nested feedback encoding, all feedback linearizable systems are hence additively approachable.

The most general theorem of this paper can be given resorting to dynamic feedback encoding. Indeed, recalling results from [16], [31], [32], we can state the following

Theorem 5: Every differentially flat system is locally additively approachable.

IV. Linear Systems

In this section we consider linear systems of type

$$\dot{x} = Fx + Gu \tag{13}$$

with $x \in \mathbb{R}^n$, $u \in U = \mathbb{R}^r$ and rank $G = r$. We start by some preliminary results characterizing the equilibrium set $\mathcal{E}$.

A. Preliminaries

Lemma 6: For a controllable linear system (13), dim $\mathcal{E} = r$.

Proof: The equilibrium equation is written as

$$0 = \left[\begin{array}{c|c} F & G \end{array}\right] \left[\begin{array}{c} x \\ \hline u \end{array}\right]. \tag{14}$$

By the PBH test, a pair (F, G) is controllable if and only if the matrix $[F - \lambda I_n \mid G]$ is full rank for all $\lambda \in \mathbb{R}$. Applying this with $\lambda = 0$, we gather directly that dim ker $\left[\begin{array}{c|c} F & G \end{array}\right] = r$. ■

Application to (13) of piecewise constant encoding of symbolic inputs (scheme *a* in fig.2) with durations $T_i = T$, $\forall i$, generates the discrete-time linear system

$$x^+ = Ax + Bu, \tag{15}$$

with

$$A = e^{FT},\ B = \left(\int_0^T e^{(T-s)F} ds\right) G.$$

Lemma 7: The equilibrium manifold of a controllable linear continuous-time system is invariant under discrete-time feedback encoding, for almost all sampling times T.

Proof: Let $\mathcal{E}$ and $\mathcal{E}'$ denote the equilibrium manifold of (13) and (15), respectively. It holds $\mathcal{E} \subseteq \mathcal{E}'$: indeed, all equilibrium pairs $(\bar{x}, \bar{u})$ for (13) are also such for (15). On the other hand, the equilibrium equation for (15) can be written as

$$0 = \left[\begin{array}{c|c} I - A & B \end{array}\right] \left[\begin{array}{c} x \\ u \end{array}\right]. \tag{16}$$

It is well known that, for almost all sampling times T, controllability of the sampled system is conserved, hence we have, again by application of the PBH lemma, that dim$\mathcal{E}' =$ dim ker $\left[\begin{array}{cc} I - A & B \end{array}\right] = r$.

The equilibrium manifold $\mathcal{E}''$ for system (15) with a linear feedback $u = Kx + w$, coincides with $\mathcal{E}'$. Indeed, writing the new equilibrium equation

$$\left[\begin{array}{c|c} I - A + BK & B \end{array}\right] \left[\begin{array}{c} x \\ u \end{array}\right] = 0, \tag{17}$$

one has that, $\forall x = \bar{x} \in \mathcal{E}'$, $u = \bar{u} - K\bar{x}$ solves (17), hence $\mathcal{E}' \subseteq \mathcal{E}''$. As controllability is not altered by state feedback, a PBH test argument as above concludes the proof. ■

A crucial observation concerning systems with drift is contained in the following lemma.

Lemma 8: For a linear system (13) with $r = 1$ and $n > 1$, it is impossible to steer the state among points in $\mathcal{E}$ while remaining in $\mathcal{E}$.

Proof: Let all solutions to (14) be written as

$$\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} N_x \\ N_u \end{bmatrix} \mu, \ \mu \in \mathbb{R}^r.$$

In order for the system's trajectory to remain in $\mathcal{E}$, it is necessary that its velocity lies in the tangent space to $\mathcal{E}$, hence that, $\forall \mu$, $\exists \mu' \in \mathbb{R}^r, u \in \mathbb{R}^r$ such that

$$\dot{x} = N_x \mu' = F N_x \mu + G u.$$

Choosing $u = N_u \mu + u'$, $u' \in \mathbb{R}^r$, one obtains the necessary condition that $\forall \mu'$, $\exists u'$ such that $N_x \mu' = G u'$, i.e. that the range space of G and of N_x must have nontrivial intersection. This condition contradicts controllability: indeed, by multiplying both sides by F we get $F N_x \mu' = F G u'$ and using $F N_x \mu' = -G N_u \mu'$, we get rank $[G|FG] = 1$. ■

The argument can be directly generalized to multi-input systems by recurring to the Brunovsky form (see e.g. [33]). Indeed, it is well known that, for a controllable system[1] $Dx = Fx + Gu$, there exist a change of coordinates S in the state space and V in the input space, and a linear feedback matrix K_0 such that $S^{-1}(F + GK_0)S = \widetilde{F}$ and $S^{-1}GV = \widetilde{G}$, with

$$\widetilde{F} = \begin{bmatrix} F_{\kappa_1} & 0 & \cdots & 0 \\ 0 & F_{\kappa_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_{\kappa_r} \end{bmatrix},$$

$$\widetilde{G} = \begin{bmatrix} g_{\kappa_1} & 0 & \cdots & 0 \\ 0 & g_{\kappa_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{\kappa_r} \end{bmatrix},$$

where κ_i denotes the i–th Kronecker control-invariant index. Accordingly, the state $\xi = S^{-1}x$ can be split in r subvectors $\xi = (\xi_1, \ldots, \xi_r)$ for which the dynamics are written as

$$\dot{\xi}_i = F_{\kappa_i} \xi_i + g_{\kappa_i} v_i', \ i = 1, \ldots, r \tag{18}$$

where $\xi_i \in \mathbb{R}^{\kappa_i}$,

$$F_{\kappa_i} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{\kappa_i \times \kappa_i},$$

$$g_{\kappa_i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\kappa_i},$$

$v_i' \in \mathbb{R}$ and $\sum_{i=1}^{r} \kappa_i = n$.

[1] the result holds for both continuous- and discrete-time systems. Accordingly, the operator D should be read as either a total derivative $Dy = \frac{d}{dt} y$ or a forward shift $Dy(k) = y(k)^+ = y(k+1)$

Assume that the state subvectors are ordered such that $\kappa_i > 1$ for $i = 1, \ldots, r'$ and $\kappa_j = 1$, $j = r' + 1, \ldots, r$. Let $\bar{\mathcal{E}} \subset \mathcal{E}$ denote the subspace corresponding to $\xi_i = 0$, $i = 1, \ldots, r'$. The dimension of $\bar{\mathcal{E}}$ is hence equal to the number of Kronecker indices equal to one. According to the above discussion, steering a system with drift from an equilibrium point $\xi(t_0) = \xi_0$ to $\xi(t_0 + \tau) = \bar{\xi} \in \mathcal{E}$ while remaining in $\mathcal{E}$ for all $t_0 \leq t \leq t_0 + \tau$, can only be achieved in the very special case that $\bar{\xi} - \xi_0 \in \bar{\mathcal{E}}$.

This observation motivates consideration of policies for *periodic* steering among equilibria, i.e. such that $\xi(t) \in \mathcal{E}$, $\forall t = t_0 + kT$, $T > 0$, $k = 0, 1, 2, \ldots$, while $\xi(t) \notin \mathcal{E}$ is allowed $\forall t \neq t_0 + kT$.

B. Periodic Additive Approachability

The linear discrete time system (15) is not additively approachable with $u \in U \subset \mathbb{Q}^r$, a discrete rational set. Indeed, if the system could be put in the form (7), then e^{FT} should be similar to the identity matrix, which cannot be the case for a controllable system with drift. Nor would the application of a simple (linear) feedback encoding such as that in scheme *c* in fig.2 help in this regard, as we would only get a system in the form (15) with $A = e^{(F+GK)T}$.

An encoding of symbolic inputs achieving periodic additive approachability with period ℓT, $1 < \ell \in \mathbb{N}$ for linear systems with drift can be conceived based on feedback encoding for the discrete-time system (15).

Theorem 9: For a controllable linear discrete-time system $x^+ = Ax + Bu$, there exists an integer $\ell > 1$ and a linear feedback encoding

$$\begin{aligned} E: \quad & \Sigma \to \mathcal{U}, \\ & \sigma_i \mapsto Kx + w_i \end{aligned}$$

with constant $K \in \mathbb{R}^{n \times n}$ and $w_i \in \mathcal{W}$, $\mathcal{W} \subset \mathbb{R}^r$ a quantized control set, such that, for all subsequences of period ℓT extracted from $x(\cdot)$, the reachable set is a lattice of arbitrarily fine mesh. In other terms, for $z(k) = x(\tau + k\ell)$, $\tau, k \in \mathbb{N}$, it holds

$$z^+ = z + \bar{H}\mu, \ \bar{H} \in \mathbb{R}^{n \times n}, \ \mu \in \mathbb{Z}^n$$

and $\forall \varepsilon$ there exists a choice of a finite $\mathcal{W}$ such that $\|\bar{H}\| < \varepsilon$.

We recall preliminarily a result which can be derived directly from [15].

Lemma 10: The reachable set of the scalar discrete time linear system $\xi^+ = \xi + v$, $\xi \in \mathbb{R}$, $v \in \mathcal{W} := \gamma W$ with $\gamma > 0$ and $W = \{0, \pm w_1, \ldots, \pm w_m\}$, $w_i \in \mathbb{N}$ with at least two elements w_i w_j coprime, is a lattice of mesh size γ.

Proof: Theorem 9.

For the controllable pair (A, B), let S, V, and K_0 be matrices such that $(S^{-1}(A + BK_0)S, S^{-1}BV)$ is in Brunovsky form (see above). In the new coordinates $\xi = S^{-1}x$ we have

$$\xi^+ = S^{-1}(A + BK_0)S\xi + S^{-1}BVv' = \widetilde{A}\xi + \widetilde{B}v'.$$

Let $v' = K_1\xi + v$, where:

- $v \in \mathcal{W} = \gamma_1 {}^1W \times \cdots \times \gamma_r {}^rW$, with ${}^kW = \{0, \pm^k w_1, \ldots, \pm^k w_{m_k}\}$, ${}^kw_j \in \mathbb{N}$ $k = 1, \ldots, r$, $j = 1, \ldots, m_k$, each ${}^kW$ including at least two coprime elements ${}^kw_i$ ${}^kw_j$;

- $K_1 \in \mathbb{R}^{r\times n}$ such that its i–th row (denoted K_{1i}) contains all zeroes except for the element in the $(\kappa_{i-1}+1)$–th column which is equal to one (recall that by definition $\kappa_0 = 0$).

Using notation as in (18), it can be easily observed that $(A_{\kappa_i} + B_{\kappa_i} K_{1i})^{\kappa_i} = I_{\kappa_i}$, the $\kappa_i \times \kappa_i$ identity matrix. Hence, if we let

$$\ell = \text{l.c.m.}\ \{\kappa_i : i = 1, ..., r\},$$

we get $\left[S^{-1}((A+BK_0)S + BVK_1)\right]^{\ell} = I_n$.

Let $\xi_i \in \mathbb{R}^{\kappa_i}$ denote the i–th component of the state vector relative to the pair $(A_{\kappa_i}, B_{\kappa_i})$. For any $\tau \in \mathbb{N}$ we have

$$\xi_i(\tau+\kappa_i) = \xi_i(\tau) + \begin{bmatrix} v_i(\tau) \\ \vdots \\ v_i(\tau+\kappa_i-1) \end{bmatrix} \qquad (19)$$

On the longer period of ℓT, we have

$$\begin{aligned} \xi_i(\tau+\ell) &= \xi_i(\tau) + \begin{bmatrix} \sum_{k=0}^{\frac{\ell}{\kappa_i}-1} v_i(\tau+k\kappa_i) \\ \vdots \\ \sum_{k=0}^{\frac{\ell}{\kappa_i}-1} v_i(\tau+\kappa_i-1+k\kappa_i) \end{bmatrix} \\ &:= \xi_i(\tau) + \bar{v}_i(\tau), \end{aligned}$$

hence

$$\xi(\tau+\ell) = \xi(\tau) + \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_r \end{bmatrix} := \xi(\tau) + \bar{v}$$

or, in the initial coordinates,

$$x(\tau+\ell) = x(\tau) + S\bar{v}.$$

In conclusion, by the linear discrete–time feedback encoding

$$\begin{aligned} E: \quad & \Sigma \to \mathcal{U}, \\ & \sigma_i \mapsto (K_0 + VK_1S^{-1})x + Vv_i \end{aligned}$$

with $v_i \in \mathcal{W}$, for all ℓ-periodic subsequences $z(k) = x(\tau + k\ell)$, it holds

$$z^+ = z + S\Gamma\mu, \ \ \mu \in \mathbb{Z}^n$$

with

$$\Gamma = \text{diag}\left(\gamma_1 I_{\kappa_1}, \cdots, \gamma_r I_{\kappa_r}\right).$$

It is also clear that, for any ε, it is possible to choose Γ such that z can be driven in a finite number of steps (multiple of ℓ) to within an ε-neighborhood of any point in $\mathbb{R}^n$. ∎

It is interesting to note that, for single-input systems, the encoding considered in theorem 9 is indeed optimal, in terms of minimizing the periodicity by which the lattice is achievable.

Proposition 11: Let the single-input discrete time linear control system be described by a pair (A, B) in Brunovsky form of dimension n. Then for all $j < n$ and $K_k \in \mathbb{R}^n$, $k = 1, \ldots, j$, $\prod_{k=1}^{j}(A + BK_k) \neq I_n$. Moreover if $j = n$ and $\prod_{k=1}^{n}(A + BK_k)^n = I_n$ then necessarily $K_1 = \cdots = K_n = [1\ 0\ \cdots\ 0]$.

Proof: Assume first $j < n$, then the first $n-j$ rows of the matrix $\prod_{k=1}^{j}(A+BK_k)$ are given by the vectors $e_{j+1}, \ldots, e_n$. Thus we deduce that the matrix $\prod_{k=1}^{j}(A+BK_k)$ can not be equal to the identity matrix for any choice of the feedback matrices K_k's.

For $j = n$, we have that the first row of the matrix $\prod_{k=1}^{n}(A + BK_k)$ is given by K_n, which implies that $K_n = e_1$. With this choice of K_n the second row is given by the vector K_{n-1} shifted by one position, i.e.

$$[(K_{n-1})_n \ (K_{n-1})_1 \ \cdots \ (K_{n-1})_{n-1}].$$

This implies that $K_{n-1} = e_1$. By recursion we obtain the thesis. ∎

However, for multi-input systems, the period of $(\text{l.c.m.}_i\, \kappa_i)\, T$ used in theorem 9 can be reduced to a minimal periodicity of $(\max_i \kappa_i)\, T$. This can be achieved by the planning algorithm described below in section IV-D.

C. Moving among equilibria

By the discrete-time feedback encoding scheme above discussed, any reachable state can be made an equilibrium state for subsequences of period ℓ of the discretized system. As it can be expected, however, in general the behaviour of the system amid such periodic samples is not specified, and may turn out to be unacceptable. Indeed, it can be easily observed that, for each κ_i–dimensional subsystem, the intersample dynamics are written as

$$\xi_i^+ = \left[\begin{array}{c|c} 0 & I_{\kappa_i-1} \\ \hline 1 & 0 \end{array}\right] \xi_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_i, \ \ i = 1, \ldots, r \qquad (20)$$

hence, within ℓ steps, each state variable takes once the values other states have at the first step. If a goal has to be reached, which is far from the origin, the intersample behaviour may have a large-span erratic behaviour.

However, recall that our main interest is to steer systems among states of equilibrium. We will show in this section that our feedback encoding scheme allows to solve this problem while keeping the system's evolution arbitrarily close to the equilibrium manifold. The proof of this property is obtained by comparing the length of the path produced by our method with that of the geodesic line joining the same end points (such shortest path not being attainable by any control law).

Notice that, in Brunovsky coordinates, $\mathcal{E}$ has a particularly simple structure. Letting $\mathbf{1}_{\kappa_i} \in \mathbb{R}^{\kappa_i}$ denote a vector with all components equal to 1, we have that for each κ_i-dimensional subsystem in (20), equilibrium states are $\bar{\xi}_i = \alpha_i \mathbf{1}_{\kappa_i}$, $\alpha_i \in \mathbb{R}$, hence

$$\mathcal{E} = \left\{\bar{\xi} | \bar{\xi} = \text{diag}\left(\alpha_1 I_{\kappa_1}, \cdots, \alpha_r I_{\kappa_r}\right) \mathbf{1}_n\right\}$$

For simplicity, consider the (worst) case of a system consisting of a single Brunovsky block, with initial state $\xi(0) \in \mathcal{E}$, and apply a sequence of $\ell = n$ controls $v(0), \ldots, v(n-1)$. Let $\xi(k)$ be the corresponding trajectory, and let P denote the polygonal through $\xi(k)$, $k = 0, \ldots n$. To estimate the length of P, consider its l_1 norm

$$l_1(P) = \sum_{k=1}^{n} \|\xi(k) - \xi(k-1)\|_1$$

A direct computation gives

$$\xi(k) = \xi(k-1) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v(0) \\ v(1)-v(0) \\ \vdots \\ v(k-1)-v(k-2) \end{pmatrix}.$$

We thus obtain

$$\begin{aligned} l_1(P) &= |v(0)| + (|v(0)| + |v(1)-v(0)|) + \cdots \\ &+ (|v(0)| + |v(1)-v(0)| + \cdots + |v(n-1)-v(n-2)|) \\ &\le \sum_{k=0}^{n-1} \big(2n-2k-1\big)|v(k)| \le \sum_{k=0}^{n-1} \big(2n-1\big)|v(k)|, \end{aligned}$$

hence we have

$$l(P) \le l_1(P) \le (2n-1)\|\underline{v}\|_1 \le \sqrt{n}(2n-1)\|\underline{v}\|, \quad (21)$$

where $l(P)$ denotes the Euclidean length of P, and $\|\underline{v}\|_1$, $\|\underline{v}\|$ ($\underline{v} = [v(0) \cdots v(n-1)]^T$) denote the geodesic distance between the initial and final points, in the 1-norm and in the Euclidean norm, respectively.

Inequality (21) applies to any path starting from $\mathcal{E}$. If we impose that the final point is also in the equilibrium manifold, we have

$$\xi(\ell) = \begin{bmatrix} v(0) \\ \vdots \\ v(n-1) \end{bmatrix} = \alpha \mathbf{1}_n$$

hence the condition

$$l(P) = l_1(P) = \|\underline{v}\|_1 = n|\alpha| = \sqrt{n}\|\underline{v}\|. \quad (22)$$

We are thus ready to prove the following:

Theorem 12: For every controllable linear system and $\varepsilon > 0$, there exists a control encoding such that the following holds. For every couple of points x_0, x_f both on the equilibrium manifold $\mathcal{E}$ there exists a path $x(\cdot)$ connecting x_0 to an ε neighborhood of x_f such that $d(x(t), \mathcal{E}) < \varepsilon$ for every t, where $d(\cdot, \mathcal{E})$ is the euclidean distance from $\mathcal{E}$.

Proof: First choose a control encoding as in Theorem 9 having as reachable set a lattice of mesh size $\varepsilon/\sqrt{n}$.
Now, for every fixed $x_0, x_f \in \mathcal{E}$ choose intermediate points $x_1, \ldots, x_N$ on $\mathcal{E}$ such that $\|x_i - x_{i-1}\| \le \varepsilon/\sqrt{n}$ for every $i = 1, \ldots, N$, and $\|x_f - x_N\| \le \varepsilon$. By the above reasoning, there exists a path P_i connecting x_{i-1} to x_i, $i = 1, \ldots, N$, such that the estimate (22) holds true. Then, we can conclude that $\|y - x_i\| < \varepsilon$ for every $y \in P_i$ and, since $x_i \in \mathcal{E}$, the theorem is proved patching together the paths P_i. ■

D. Planning algorithms and specification complexity

Based on the above results, a planning strategy for steering among equilibria can be obtained at once, which consists in using constant control values for a large enough number of steps. Indeed we have

Proposition 13: The application of a constant control $v_i(\tau + k) = \hat{v}_i$, $i = 1, \ldots, r$, for $0 \le k \le \ell - 1$, steers system (20) from $\xi(\tau) \in \mathcal{E}$ to $\xi(\tau + \ell) = \xi(\tau) + \mathrm{diag}\left((\frac{\ell}{\kappa_1}\hat{v}_1)I_{\kappa_1}, \cdots, (\frac{\ell}{\kappa_r}\hat{v}_r)I_{\kappa_r}\right)\mathbf{1}_n \in \mathcal{E}$.

Notice however that a planner based on the straightforward application of this proposition could lead to an inefficient solution, as the size of the mesh for the κ_i–dimensional susbsystem would be increased by a factor $\frac{\ell}{\kappa_i}$.

We now provide explicitly a more efficient method to steer from an arbitrary state $x \in \mathbb{R}^n$ to within an ε-neighborhood of a given goal state $x + \delta \in \mathbb{R}^n$ (x and δ not necessarily in $\mathcal{E}$).

1) Compute the desired displacement in Brunovsky coordinates $\Delta = S^{-1}\delta$, and let $\Delta_i \in \mathbb{R}^{\kappa_i}$, $i = 1, \ldots, r$ denote the desired displacement for the i–th subsystem;
2) Compute the lattice mesh size in Brunovsky coordinates $\gamma_i = \frac{2\varepsilon}{\|\zeta_i\|}$, where
$$\begin{bmatrix} \zeta_1 & \zeta_2 \cdots & \zeta_r \end{bmatrix} = S \begin{bmatrix} \mathbf{1}_{\kappa_1} & 0 & \cdots & 0 \\ 0 & \mathbf{1}_{\kappa_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathbf{1}_{\kappa_r} \end{bmatrix};$$
3) Find $\bar{\Delta}_i$, the nearest point to Δ_i on the lattice generated by $\gamma_i\, {}^iW$ and centered at $\xi_i = (S^{-1}x)_i$.
4) For each $i = 1, \ldots, r$, let the quantized control set be ${}^iW = \{0, \pm {}^iw_1, \ldots, \pm {}^iw_{m_i}\}$, ${}^iw_j \in \mathbb{N}$, and denote by ${}^iU$ the vector $[{}^iw_0\ {}^iw_1\ \cdots\ {}^iw_{m_i}]$, where ${}^iw_0 = 0$. Write
$$\bar{\Delta}_i = \gamma_i\, {}^iC\, {}^iU \quad (23)$$
where ${}^iC$ is a matrix in $\mathbb{Z}^{\kappa_i \times m_i + 1}$ with components ${}^iC_{h,j+1} = {}^ic_{h,j}$, $h = 1, \ldots, \kappa_i$, $j = 0, \ldots, m_i$. Each element ${}^ic_{h,j}$ of ${}^iC$ describes the number of times that the control ${}^iw_j$ has to be used to steer the h–th component of ξ_i.
5) Find integers ${}^ic_{h,j}$, $h = 1, \ldots, \kappa_i$, $j = 1, \ldots, m_i$ solving the system of diophantine equations (23), and find the smallest integers ${}^ic_{h,0}$ such that, $\forall h$, $\sum_{j=0}^{m_i} |{}^ic_{h,j}| := N_i$. $N_i\kappa_i$ is thus a number of steps sufficient to steer the i–th subsystem;
6) (Optional) Among all solutions of (23), find the one which minimizes $\max_{h=1}^{\kappa_i} \sum_{j=1}^{m_i} |\,{}^ic_{h,j}| := \hat{N}_i$. Notice that $\hat{N}_i\kappa_i$ is the minimum length of a string of symbols in ${}^iW$ obtaining the goal. However, no polynomila-time algorithm is known for such optimization;
7) Let $N_\kappa^\star = \max_i N_i\kappa_i$, and $i^\star$ the corresponding index. Then, for all $i = 1, \ldots, r$ $i \ne i^\star$, compute $\tilde{\Delta}_i = (\widetilde{A}_i)^{-r_i}(\xi + \bar{\Delta}_i) - \xi$ with $r_i = N_\kappa^\star - N_i\kappa_i$. Repeat steps 4) and 5) with the new $\tilde{\Delta}_i$.

Remark 2: Notice that, since the matrix $(\widetilde{A}_i)^{-r_i}$ is elementary (a permutation of rows and columns of the identity matrix), it has the only effect of exchanging the components of $(\xi_i + \bar{\Delta}_i)$. Therefore, $\xi_i + \tilde{\Delta}_i$ belongs to the same lattice, and it can be reached in $N_i\kappa_i$ steps.

An upper bound on the specification complexity of a generic plan is thus given by $N_\kappa^\star\, r \log_2 (1 + 2\sum_{i=1}^r m_i)$ when the input sets ${}^iW$ are disjoint, or $N_\kappa^\star\, r \log_2(1 + 2m)$ if ${}^iW = {}^jW$, $i, j = 1, \ldots, r$. The choice of section II-B for all input sets provides plans with low $N_\kappa^\star$ to join any two points in a hypercube of size M, as required in problem Π.

Encoding these plans by their run-length reduces their specification complexity. Indeed we have

Proposition 14: The specification complexity of a plan from x to $x+\delta$ according to the above algorithm encoded by run-length, is of order

$$\mathcal{C} \sim n(1+m)\max_i (\log_2 N_i) . \tag{24}$$

Proof: We use the notation established in (23), whereby ${}^iU$ denotes an ordered set of (m_i+1) symbols, and associate the run count ${}^iC_{h,j+1}$ to symbol ${}^iw_j$ if ${}^iC_{h,j+1} \geq 0$, or to symbol $-{}^iw_j$ if ${}^iC_{h,j+1} < 0$. The plan is thus completely described by the r matrices ${}^iC \in \mathbb{Z}^{\kappa_i \times (m_i+1)}$. For specifying them, a total of $\sum_{i=1}^r \kappa_i(m_i+1)$ integers in $[-N_i, N_i]$ is needed. This amounts to $\sum_{i=1}^r (m_i+1)\kappa_i \left(1+\lceil \log_2(N_i+1)\rceil\right)$ bits (in sign-magnitude representation). If the same set U of symbols is used for all inputs, we can bound the bit number by $n(m+1)\left(1+\lceil \log_2(1+\max_i N_i)\rceil\right)$, hence the statement. ■

This result can be further improved for plans among equilibrium configurations using optimal input sets as described in section II-B. The following claim holds for $m \leq 4$, and relies on conjecture in [24] otherwise:

Theorem 15: The specification complexity of a solution P to problem Π with $\mathcal{M}$ the hypercube of half-size M in the r-dimensional equilibrium subspace $\mathcal{E}$ of X, is of order

$$\mathcal{C}(P) \sim \alpha r \log_2\left(\frac{M}{\varepsilon}\right),$$

with $\alpha = \left(\frac{2M}{\varepsilon}\right)^r$.

Proof: Observe that, when moving among equilibria, the displacement also belongs to $\mathcal{E}$, and is thus described by a vector with equal components. Hence, each matrix ${}^iC$ has identical rows, and can be specified by $(m+1)(1+\lceil \log_2(N_i+1)\rceil)$ bits. Furthermore in this case, plan lengths N_i can be equalized to $N = \max_i N_i$ by simply padding shorter control sequences with zeroes. Recalling the estimate (11), if ε is the mesh size of the reachable set and M is the maximum distance reachable in N steps, we can write

$$m \sim \frac{\log_2\left(\frac{M}{\varepsilon}\right)}{\log_2 N}.$$

Hence, being P comprised of α different point-to-point plans, we have

$$\begin{aligned}\mathcal{C}(P) &\sim \alpha r(m+1)(1+\log_2 N)\\ &\sim \alpha r\left(\frac{\log_2\left(\frac{M}{\varepsilon}\right)}{\log_2 N}+1\right)(1+\log_2 N)\\ &\sim \alpha r\left(\log_2\left(\frac{M}{\varepsilon}\right)+\log_2 N\right)\end{aligned}$$

Having observed in section II-B that $N = o(M)$, we finally have

$$\mathcal{C}(P) \sim \alpha r \log_2\left(\frac{M}{\varepsilon}\right).$$

■

The explicit construction of a procedure to decode plan specifications ${}^iC$ into a string of control inputs ${}^iV$ for the i–th channel is finally described in Matlab-like code:

```
C=iC;
iV=[];
while (C ~= 0)
  for h=1:κi,
    j=1;
    while C(h,j) == 0, j=j+1; end
    iV = cat (iV, sign(C(h,j))*iwj);
    C(h,j) = C(h,j)-sign(C(h,j));
  end
end
```

V. Examples

A. Example 1: Linear System

As a first example, consider the problem of planning rest-to-rest motions of a cart, actuated by horizontal forces, with an inverted pendulum attached, which should start and return to the upright equilibrium. It is desired to plan motions to reach all points in an interval $\pm M$ of the horizontal line, with resolution ε. Each plan should be comprised of sequences of finite length N using $2m+1$ different symbols.

Consider a linearized model of the system as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & \frac{-mg}{M} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & \frac{(M+m)g}{lM} & 0\end{bmatrix} x + \begin{bmatrix} 0\\ \frac{1}{M}\\ 0\\ \frac{-1}{lM}\end{bmatrix} u, \tag{25}$$

where x_1, x_2 is the cart position and velocity, x_3, x_4 the pendulum angle and its rate, and the input u is the horizontal force applied to the cart. Numerical values will be used as $l = 1$m for the pendulum's length, $g = 9.81$ m/s^2 for gravity acceleration, $M = 20$Kg and $m = 5$Kg for cart and pendulum masses, respectively. The equilibrium manifold is $\mathcal{E} = \{x \in \mathbb{R}^4 | x_1 = \alpha \in \mathbb{R}, x_2 = x_3 = x_4 = 0\}$. Apply the discrete-time feeedback encoding of fig. 3-d, with unit sampling time $T = 1$s. Accordingly, the sampled system is

$$x^+ = Ax + Bu =$$

$$= \begin{bmatrix} 1 & 1 & -3.12 & -0.74\\ 0 & 1 & -11.60 & -3.12\\ 0 & 0 & 16.60 & 4.73\\ 0 & 0 & 58.03 & 16.60\end{bmatrix} x + \begin{bmatrix} 0.03\\ 0.08\\ -0.06\\ -0.23\end{bmatrix} u,$$

and reachability is preserved. Let S be a change of coordinates such that $(S^{-1}AS, S^{-1}B)$ is in control canonical form. In the new coordinates, the equilibrium set is $\mathcal{E} = \{\beta \mathbf{1}_4, \beta \in \mathbb{R}\}$. For corresponding equilibria in the two coordinate systems it holds $\alpha = S_F\beta$, with scale factor $S_F = \|S\mathbf{1}_4\| = 1.24$.

To obtain the required resolution of ε, choose a mesh size $\gamma = \frac{2\varepsilon}{(S_F)}$ in S coordinates. To this purpose, W can be chosen to be any finite sets of integers, such that at least two of its elements are coprime, and inputs scaled as $v \in \gamma W$.

Consider for instance the use of a rather sparse control set with $m = 3$ (cf. table I). Taking e.g. $\varepsilon = 0.01$m and $M = 0.75$m, one gets $\gamma = 0.0161$, hence $M/\gamma \approx 46.5$. From table I for $m = 3$, we get $N = 5$. Observe that the actual execution of the plan takes $n = 4$ times N sampling instants, because of the dimension of the state space.

If a rest-to-rest displacement of 0.65m on the horizontal line is specified, the closest point on the reachable lattice is given by $\delta = (64\varepsilon, 0, 0, 0)$. In control canonical form we can write $32\gamma\mathbf{1}_4 = \gamma CU$, where $U = (0, 11, 14, 15)$ and C is a matrix

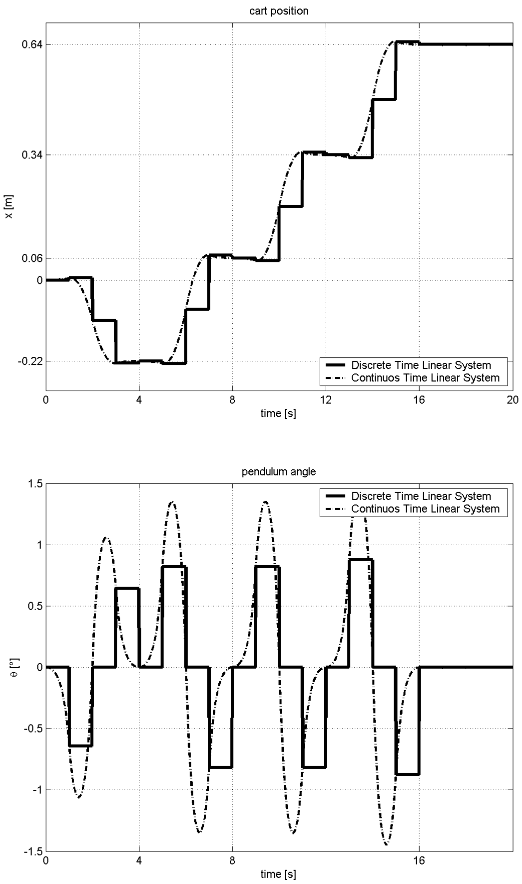


Fig. 4. Planned motions of the cart (above) and of the pendulum (below) in example 1

in $\mathbb{Z}^{4\times 4}$ obtained by the $\min\max$ problem in the algorithm of section IV-D as

$$C = \begin{bmatrix} 0 & -1 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix}.$$

The motion of the cart and the pendulum corresponding to such plan are shown in fig. 4.

B. Example 2: Nonlinear System

By using the nested feedback encoding of fig. 3, it is possible to directly apply the proposed steering method to the substantially wider class of systems which are dynamic feedback–equivalent to linear systems. In this example, we illustrate the power of the proposed method by solving the steering problem for an example in the class of underactuated mechanical systems, which have attracted wide attention in the recent literature (see e.g. [34]).

In particular, we consider the class of underactuated mechanisms identified as "$(n-1)X_a - R_u$ planar robots", i.e. mechanisms having $n-1$ active joints of any type, and a passive rotational joint. In order to simplify the model analysis and control design, it is convenient to use a specific set of generalized coordinates. In particular, let $q = (q_1, ..., q_{n-3}, x, y, \theta) = (q_a, \theta)$ where (x, y) are the cartesian coordinates of the base of the last link. Assuming motion in a horizontal plane (or zero gravity), the dynamic model takes on the partitioned form

$$\begin{bmatrix} \multicolumn{3}{c}{\multirow{3}{*}{$B_a(q_a)$}} & 0_{(n-3)\times 1} \\ & & & -m_n d_n s_\theta \\ & & & m_n d_n c_\theta \\ 0_{1\times(n-3)} & -m_n d_n s_\theta & m_n d_n c_\theta & I_n + m_n d_n^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{\theta} \end{bmatrix} + \\ + \begin{bmatrix} c_a(q, \dot{q}) \\ 0 \end{bmatrix} = \begin{bmatrix} F_a \\ 0 \end{bmatrix} \quad (26)$$

where $F_a = (F_1, ..., F_{n-3}, F_x, F_y)$ are the generalized forces performing work on the q_a coordinates, $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. For the n–th link, I_n, m_n and d_n are the baricentral inertia, the mass and the distance of the center of mass from its base. We assume that the robot arm has at least two active joints ($n \geq 3$), so that forces F_x and F_y can be independently assigned. By the virtual work principle, we can recover the original forces $\bar{\tau}$ acting on the joints from F_a as

$$\bar{\tau} = \begin{bmatrix} \bar{\tau}_1 \\ \vdots \\ \bar{\tau}_{n-1} \end{bmatrix} = \begin{bmatrix} I_{(n-3)\times(n-3)} \\ 0_{2\times(n-3)} \end{bmatrix} \begin{bmatrix} F_1 \\ \vdots \\ F_{n-3} \end{bmatrix} + \\ + J^T(q_1, ..., q_{n-1}) \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

where J is the Jacobian matrix $2 \times (n-1)$ of the direct kinematics function k.

In order to make the analysis independent from the nature of the $n-1$ active joints, the relative dynamics in (26) can be linearized via a globally defined partial static feedback, thus reducing them to a chain of two integrators per actuated joint. As in the computed torque method, such partially linearizing static feedback is $F_a = \hat{B}_a(q)a + c_a(q, \dot{q})$ where

$$\hat{B}_a(q) = B_a(q_a) - \\ - \frac{m_n^2 d_n^2}{I_n + m_n d_n^2} \begin{bmatrix} 0_{(n-3)\times(n-3)} & \multicolumn{2}{c}{0_{(n-3)\times 2}} \\ 0_{2\times(n-3)} & \begin{matrix} s_\theta^2 \\ -s_\theta c_\theta \end{matrix} & \begin{matrix} -s_\theta c_\theta \\ c_\theta^2 \end{matrix} \end{bmatrix}.$$

The closed loop system then becomes $\ddot{q}_1 = a_1$, ..., $q_{n-3} = a_{n-3}$, $\ddot{x} = a_x$, $\ddot{y} = a_y$, $\ddot{\theta} = \frac{1}{K}(s_\theta a_x - c_\theta a_y)$ where $K_{CP} = \frac{I_n + m_n d_n^2}{m_n d_n}$ is the distance of the center of percussion of the last link from its base. Assuming uniform mass distribution we have $K_{CP} = 2/3\, l_n$, where l_n is the length of the nth link. The dynamics of the coordinates q_i, $i = 1, ..., n-3$ are completely decoupled from the dynamics of the remaining coordinates (x, y, θ). Therefore, we will henceforth only consider the case $n = 3$.

Following [34], we choose the cartesian coordinates of the center of percussion as the system's (flat) outputs:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + K_{CP} \begin{bmatrix} c_\theta \\ s_\theta \end{bmatrix}. \quad (27)$$

Differentiation of equation (27) yields

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + K_{CP}\dot{\theta} \begin{bmatrix} -s_\theta \\ c_\theta \end{bmatrix} \tag{28}$$

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} c_\theta^2 & s_\theta c_\theta \\ s_\theta c_\theta & s_\theta^2 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} - R(\theta) \begin{bmatrix} K_{CP}\dot{\theta}^2 \\ 0 \end{bmatrix} \tag{29}$$

where $R(\theta)$ is the matrix associated to a planar rotation of an angle θ. Being the matrix multiplying the acceleration vector (a_x, a_y) singular, dynamic feedback must be considered. Define an invertible feedback transformation

$$\begin{bmatrix} a_x \\ a_y \end{bmatrix} = R(\theta) \begin{bmatrix} \chi + K_{CP}\dot{\theta}^2 \\ \psi_2 \end{bmatrix} \tag{30}$$

where χ and ψ_2 are two auxiliary variables. As a result we have

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = R(\theta) \begin{bmatrix} \chi \\ 0 \end{bmatrix}. \tag{31}$$

By adding two integrators on the first auxiliary variable, namely setting $\dot{\chi} = \iota$, $\dot{\iota} = \psi_1$, and differentiating equation (29), we get

$$\begin{bmatrix} y_1^{(3)} \\ y_2^{(3)} \end{bmatrix} = R(\theta) \begin{bmatrix} \nu \\ \chi\dot{\theta} \end{bmatrix} \tag{32}$$

$$\begin{bmatrix} y_1^{(4)} \\ y_2^{(4)} \end{bmatrix} = R(\theta) \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -\frac{\xi}{K_{CP}} \end{bmatrix} \psi + \begin{bmatrix} -\chi\ddot{\theta}^2 \\ 2\iota\dot{\theta} \end{bmatrix} \right\} =$$

$$= R(\theta) \left\{ P(\theta,\chi)\sigma + q(\theta,\dot{\theta},\chi,\iota) \right\}, \tag{33}$$

with $\psi = (\psi_1, \psi_2)$ the new input variables.

Under the assumption that $P(\theta,\chi)$ is nonsingular or, equivalently, away from $\chi \neq 0$, the inversion-based control $\psi = P^{-1}(\theta,\chi)(R^T(\theta)v - q(\theta,\dot{\theta},\chi,\iota))$, with $v = (v_1, v_2)$ as new input vector, yields two decoupled chains of four input-output integrators. The assumption means that pure rotational motion around the center of percussion is not allowed. Since the relative degree is 8 $(4+4)$, the dimension of the robot state is 6 $(x, \dot{x}, y, \dot{y}, \theta, \dot{\theta})$ and the dimension of the compensator is 2 (χ, ι), exact linearization has been achieved.

For the integrators of the compensator, we use zero initial conditions. In order to satisfy the assumption, we impose a nonzero linear acceleration of the last link along its axis.

The dynamics of the system after the dynamic feedback linearization are written as $y_1^{(4)} = v_1$, $y_2^{(4)} = v_2$. Choosing a sample time $t = 1s$ we obtain the following discrete time linear system:

$$x_i^+ = Ax_i + Bv_i =$$

$$= \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_i + \begin{bmatrix} \frac{1}{24} \\ \frac{1}{6} \\ \frac{1}{2} \\ 1 \end{bmatrix} v_i$$

where $x_i = \left(y_i,\ y_i^{(1)},\ y_i^{(2)},\ y_i^{(3)}\right)$, $i = 1,2$. Being each subsystem controllable, there exist S such that $(S^{-1}AS, S^{-1}B)$ is in control canonical form. For each subsystem in control canonical form, the set of equilibria is given by $\{\alpha\mathbf{1}_4 \in \mathbb{R}^4 : \alpha \in \mathbb{R}\}$. Then, in the initial coordinates, the set of equilibria is given by $\{\alpha S\mathbf{1}_4 \in \mathbb{R}^4 : \alpha \in \mathbb{R}\}$. For a given $\alpha \in \mathbb{R}$ we obtain the equilibrium $\alpha\mathbf{1}_4$ for the control canonical form and the equilibrium $(\alpha, 0, 0, 0)$ for the original subsystem, hence a constant position of the considered coordinate of the center of percussion. The scale factor is 1 in this case.

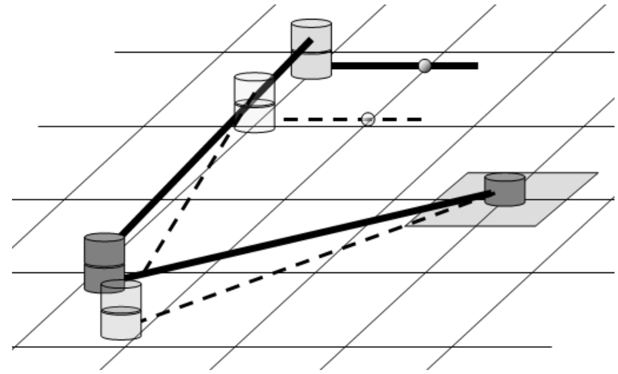


Fig. 5. An underactuated robot arm of type $2R_a - R_u$ used in example 2: the given initial and final configurations are shown by dashed and solid lines, respectively.

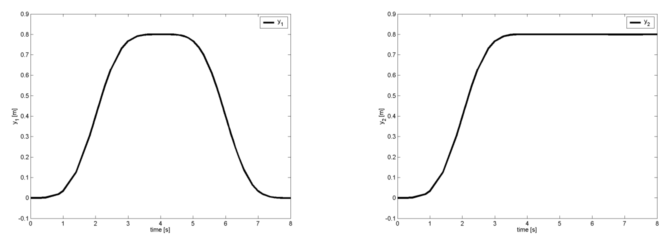


Fig. 6. Coordinates y_1 (left) and y_2 (right) of the center of percussion

To obtain a reachable lattice of size $\gamma_1, \gamma_2 > 0$, ${}^1W$, ${}^2W$ can be chosen to be any finite sets of integers, such that at least two of its elements are coprime, and and inputs scaled as ${}^i v \in \gamma_i\, {}^i W$, $i = 1,2$.

Given an initial robot pose $(y_1, y_2, \theta) = (0,0,0)$, consider three maneuvers: translation along the x axis, translation along the straight line $y = x$ and translation along the y axis. The first one can be achieved with a single symbol w applied on the input ${}^1v$ for $n = 4$ periods. The second maneuver is similar to the first one: we apply the previous command on the two inputs ${}^1v$ and ${}^2v$ for $n = 4$ periods. We can split the third maneuver in two maneuvers of the previous types.

Initial and final positions of a 3R robot are shown in fig. 5. Simulations were performed setting $l1 = l2 = 3m$, $K_{CP} = 1m$, $T = 1s$, and $w = 0.8\text{m/s}^2$. Fig. 6 shows the coordinates of the Center of Percussion of the last link while fig. 7 shows the angles of the active joints and the orientation of the last passive link, respectively.

VI. CONCLUSIONS

In this paper, we have described methods for steering complex dynamical systems by signals with finite-length descriptions. Of particular relevance are results on reducing the specification and computational complexity of palnning problems.

Systems tractable by symbolic control under encoding include all controllable linear systems, nilpotent driftless nonlin-

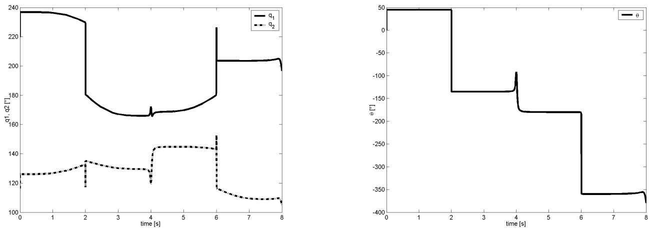


Fig. 7. Active joints angles (left) and orientation of the last passive link (right)

ear systems and (dynamically) feedback-linearizable systems. It seems fair to affirm that few practically interesting classes of controllable systems remain outside the scope of application of the presented methods — among which, most notably, are non-differentially flat systems.

Many other open problems remain open in order to fully exploit the potential of symbolic control. A limitation of our current approach is that we assume that a flat, linearizing output to be available, as well as state measurements. Connections to state observers in planning are unexplored at this stage. As mentioned in the introduction, an advantage of our planning method is that it can be computed and communicated very efficiently, hence it can be conceivably used as a feed-forward block to generate and update in real-time reference trajectories to be tracked by a complex remote system. In such an application, however, the joint stability of the feed-forward and feedback blocks would need thorough investigation.

APPENDIX I

Lemma 16: Systems in strictly triangular form (12) are additively approachable.

Proof: Consider a control encoding $\Sigma \to \mathcal{U}$, $\sigma_i \mapsto u_i$, with $\mathcal{U}$ finite set, of cardinality N_p, of constant input quanta $u_i : [0, T] \to \mathbb{R}^{n_p}$. Let $\mathcal{U}$ be symmetric, i.e. include all inverses $\bar{u}_i = -u_i$, and let $\bar{\sigma}_i \mapsto \bar{u}_i$. Define a *commutator sequence* as $[\sigma_i, \sigma_j] = \sigma_i\sigma_j\bar{\sigma}_i\bar{\sigma}_j$. By (12), we immediately get that, upon application of any commutator sequence, $x_p(t+4T) = x_p(t)$.

Let $\Omega_p := \Sigma$, and define recursively $\Omega_{k-1} = [\Omega_k, \Omega_k]$ to be the set of all commutator sequences built on Ω_k. A flag of words sets (not sub-languages) $\Omega_p \supset \Omega_{p-1} \supset \cdots \supset \Omega_1$ is thus obtained such that, for words in Ω_i, $i < p$, all state variables $(x_p, x_{p-1}, \ldots, x_{i+1})$ undergo a closed path in $\mathbb{R}^{n_{i+1}+\cdots+n_p}$. Our strategy is to use the sub-language generated by Ω_i in backward recursion to move the coordinate x_i. Let N_i denote the cardinality of Ω_i. By construction, we have $N_{i-1} = N_i(N_i - 1)/2$. To each word of $\omega_i^j \in \Omega_i$, $j = 1, \ldots, N_i$, there corresponds a quantum displacement vector $h(\omega_i^j) \in \mathbb{R}^{n_i}$ providing a net motion on x_i at intervals of $4^{p-i}T$. Hence, the action of words in Ω_i is additive on $\mathbb{R}^{n_i}$. Let $Q_i = \{h(\omega_i^j),\ \omega_i^j \in \Omega_i\}$ denote the set of quantum displacements. If $N_i \geq n_i$, then, in generic hypothesis, we can assume that $\text{span}(Q) = \mathbb{R}^{n_i}$.

Notice that Q_p is given by $\{T\sum_{i=1}^{p} g_p^i u_i, u = [u_1, \ldots, u_p] \in \mathcal{U}\}$. If $\mathcal{U}$ is formed by rational constant control quanta and $g_p^i \in \mathbb{Q}$, then the group generated by Q_p is a lattice Λ_p whose mesh size can be made as small as desired by scaling control values in $\mathcal{U}$.
Otherwise, Q_p, generically, generates a dense subset of $\mathbb{R}^{n_p}$. In this case, we can choose a symmetric subset $\widetilde{\mathcal{U}} \subset \mathcal{U}$ of cardinality $2n_p$ such that n_p of the corresponding elements of Q_p are linearly independent. Then $\widetilde{\mathcal{U}}$ generates a lattice and, again, we can decide the mesh size by rescaling controls.
For the other components x_i we can reason similarly checking if the displacements in Q_i are rationals, otherwise selecting suitable subsets of words.

■

ACKNOWLEDGMENTS

Authors would like to acknowledge Andrea Biondi for his help in working out the two examples and in translating the algorithm into functioning code. Thanks also to Adriano Fagiolini and Luca Greco for their careful reading of the draft and useful discussions.

REFERENCES

[1] M. J. van Nieuwstadt and R. M. Murray, “Real time trajectory generation for differentially flat systems,” *Int’l. J. Robust & Nonlinear Control*, vol. 8, no. 11, pp. 995–1020, 1998.
[2] F. Bullo and K. M. Lynch, “Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems,” *IEEE Trans. on Robotics and Automation*, vol. 17, no. 4, pp. 402–412, 2001.
[3] Emilio Frazzoli, Munther A. Dahleh, and Eric Feron, “Real-time motion planning for agile autonomous vehicles,” in *Proc. AIAA Guidance, Navigation, and Control Conference and Exhibit*, August 2000, pp. 1–11.
[4] E. Frazzoli, M. A. Dahleh, and E. Feron, “Maneuver-based motion planning for nonlinear systems with symmetries,” *IEEE Trans. on Robotics*, 2005.
[5] P. Tabuada and G. J. Pappas, “Hierarchical trajectory generation for a class of nonlinear systems,” *Automatica*, vol. 41, no. 4, pp. 701–708, 2005.
[6] M. Okada and Y. Nakamura, “Polynomial design of dynamics-based infomration processing system,” in *Control Problems in Robotics*, A. Bicchi, H. Christensen, and D. Prattichizzo, Eds., number 4 in STAR, pp. 91–104. Springer-Verlag, 2003.
[7] V. Manikonda, P. S. Krishnaprasad, and J. Hendler, “A motion description language and hybrid architecture for motion planning with nonholonomic robots,” in *Proc. Int. Conf. on Robotics and Automation*, 1995.
[8] M. Egerstedt, “Motion description languages for multimodal control in robotics,” in *Control Problems in Robotics*, A. Bicchi, H. Christensen, and D. Prattichizzo, Eds., number 4 in STAR, pp. 75–89. Springer-Verlag, 2003.
[9] R. Brockett, “On the computer control of movement,” in *Proc. IEEE Conf. on Robotics and Automation*, April 1988, pp. 534–540.
[10] P. Tabuada and G. J. Pappas, “Bisimilar control affine systems,” *Systems & Control Letters*, vol. 52, no. 1, pp. 49–58, 2004.
[11] Paulo Tabuada, *Hybrid Systems: Computation and Control*, vol. 3414 of *Lecture Notes in Computer Science*, chapter Sensor/actuator abstractions for symbolic embedded control design, pp. 640–654, Springer, 2005.
[12] Magnus Egerstedt and Roger W. Brockett, “Feedback can reduce the specification complexity of motor programs,” *IEEE Trans. on Automatic Control*, vol. 48, no. 2, pp. 213–223, 2003.
[13] Y. Anzai, “A note on reachability of discrete–time quantized control systems,” *IEEE Trans. on Automatic Control*, vol. 19, no. 5, pp. 575–577, 1974.
[14] Y. Chitour and B. Piccoli, “Controllability for discrete systems with a finite control set,” *Math. Control Signals Systems*, vol. 14, no. 2, pp. 173–193, 2001.
[15] A. Bicchi, A. Marigo, and B. Piccoli, “On the reachability of quantized control systems,” *IEEE Trans. on Automatic Control*, vol. 47, no. 4, pp. 546–563, April 2002.
[16] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, “Flatness and defect of nonlinear systems: Introductory theory and examples,” *Int. J. of Control*, vol. 61, pp. 1327–1361, 1995.

[17] S. M. LaValle, *Planning Algorithms*, Cambridge University Press (also available at http://msl.cs.uiuc.edu/planning/), 2005, To be published.

[18] G. Heinzinger, P. Jacobs, J. Canny, and B. Paden, "Time-optimal trajectories for a robot manipulator: a provably good approximation algorithm," in *Proc. IEEE Int. Conf. on Robotics and Automation*, 1990, pp. 150–156.

[19] B. R. Donald and P. G. Xavier, "Provably good approximation algorithms for optimal kinodynamic planning for cartesian robots and open chain manipulators," *Algorithmica*, vol. 14, no. 6, pp. 480–530, 1995.

[20] S.M. LaValle and J.J. Kuffner, "Randomized kinodynamic planning," *Int. Jour. of Robotics Research*, vol. 20, no. 5, pp. 378–400, 2001.

[21] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley Interscience Publ., 1986.

[22] L. A. Wolsey, *Integer Programming*, Wiley Interscience Publ., 1998.

[23] Richard K. Guy, *Unsolved problems in number theory*, Springer Verlag, 1994.

[24] A. Marigo, "Time optimal quantized controls," in *Internal Report*, 2004.

[25] A.V. Sarichev and H. Nijmeier, "Extremal controls for chained systems," *Journal of Dynamical Control Systems*, vol. 2, pp. 503–527, 1996.

[26] A.W. Divelbiss and J. Wen, "Nonholonomic path planning with inequality constraints," in *Proc. IEEE Int. Conf. on Decision and Control*, 1993, pp. 2712–2717.

[27] D. Tilbury, R.M. Murray, and S.S.Sastry, "Trajectory generation for the n traile problem using goursat normal form," *IEEE Trans. on Automatic Control*, vol. 40, no. 5, pp. 802–819, 1995.

[28] P. Rouchon, M. Fliess, J. Lévine, and P. Martin, "Flatness, motion planning, and trailer systems," in *Proc. IEEE Int. Conf. on Decision and Control*, 1993, pp. 2700–2705.

[29] O.J. Sordalen, "Conversion of the kinematics of a car with n trailers into a chained form," in *Proc. IEEE Int. Conf. on Robotics and Automation*, 1993, pp. 382–387.

[30] M. Kawski, "Nilpotent lie algebras of vectorfields," *Journal für die Reine und Angewandte Mathematik*, vol. 388, pp. 1–17, 1988.

[31] B. Charlet, J. Levine, and R. Marino, "Sufficient conditions for dynamic state feedback linearization," *SIAM J. Control*, vol. 29, no. 1, pp. 38–57, 1991.

[32] E. Aranda-Bricaire, C. H. Moog, and J.-B. Pomet, "A linear algebraic framework for dynamic feedback linearization," *IEEE Trans. on Automatic Control*, vol. 40, no. 1, pp. 127–132, 1995.

[33] E. Sontag, "Control of systems without drift via generic loops," *IEEE Trans. on Automatic Control*, vol. 40, no. 7, pp. 1210–1219, 1995.

[34] A. De Luca and G. Oriolo, "Trajectory planning and control for planar robots with passive last joint," *The International Journal of Robotics Research*, vol. 21, no. 5-6, pp. 575–590, 2002.

Antonio Bicchi graduated at the University of Bologna in 1988 and was a postdoc scholar at M.I.T. A.I. Lab in 1988–1990. He is currently Professor of Automatic Control in the Department of Electrical Systems and Automation (DSEA) of the University of Pisa, and Director of the Interdepartmental Research Center "E. Piaggio", University of Pisa. He has served IEEE and other professional societies and journals in several capacities, and is currently an IEEE Fellow and Vice President of IEEE Robotics and Automation Society. He has published more than 200 refereed papers. His main research interests are in the control of complex systems, including hybrid logic and dynamic systems, and in robotics at large.

Alessia Marigo Alessia Marigo received the Laurea degree in mathematics from the University of Pisa, Italy, and the Ph.D. degree in robotics from the University of Genova, Italy, in 1994 and 1999, respectively. From 1995 to 1999, she worked on controllability and motion planning, both for nonholonomic systems and for discrete systems with quantized controls at the Interdepartmental Research Center "E. Piaggio", University of Pisa. From 1999 to 2001, she has been with the group of Geometric Control Theory and Applications of the International School for Advanced Studies (SISSA/ISAS), Trieste, Italy. From 2002 she is at the Departement of Mathematics of the University of Rome "La Sapienza". Her current research concerns nilpotent approximations, systems with quantized controls and optimal control of traffic flows on telecommunication networks.

Benedetto Piccoli Benedetto Piccoli received the Laurea degree (cum laude) from the University of Padova, Italy, and the Ph.D. degree from the International School for Advanced Studies (SISSA/ISAS), Trieste, Italy, in 1991 and 1994, respectively. Currently, he is research director at the Istituto per le Applicazioni del Calcolo "Mauro Picone" of the C.N.R. He has published more than 80 papers in journals, books, and refereed conferences. He is editor in chief of Networks and Heterogeneous Media and he is in the editorial board of SIAM Journal on Control and Optimization, Journal of Dynamical and Control Systems and ESAIM Control Optimization and Calculus of Variation. His main research interests are in the field of optimal control, hybrid and quantized systems and control of traffic flows on networks.