QUANTIZED CONTROL SYSTEMS AND DISCRETE NONHOLONOMY

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Abstract: In this paper we study control systems whose input sets are quantized, and in particular finite or countable but nowhere dense. We specifically focus on problems relating to the structure of the reachable set of such systems, which may turn out to be either dense or discrete. We report results on the rechable set of linear quantized systems, and on a particular but interesting class of nonlinear systems, forming the discrete counterpart of driftless nonholonomic continuous systems. Implications and open problems in the analysis and synthesis of quantized control systems are addressed.

Keywords: Quantized control systems, Nonholonomic systems, Hybrid systems.

1. INTRODUCTION

In this paper we consider systems of the type

$$x^+ = g(x,u),\ x \in \mathbb{R}^n,\ u \in U \subset \mathbb{R}^m \qquad (1)$$

where the input set, U, is quantized, i.e. finite or numerable but nowhere dense in $\mathbb{R}^m$. Quantized control systems arise in a number of applications because of many physical phenomena or technological constraints. In the control literature, quantization of inputs has been considered mainly as due to D/A conversion, and mostly regarded as a disturbance to be rejected (Bertram (1958); Slaughter (1964); Delchamps (1990)). Typical results in this spirit are those provided by Hou *et al.* (1997), who show how a nonlinear system with quantized feedback, whose linear approximation (without quantization) has an asymptotically stable solution, has uniformly ultimately bounded solutions; and how such bounds can be made small at will by refinig quantization sufficiently.

More recently, some attention has been focused on quantized control systems as specific models of hierarchically organized systems with interaction between continuous dynamics and logic (Wong and Brockett (1999); Elia and Mitter (1999)). In these papers, quantization of inputs is regarded as a fundamental characteristic of systems where the resources for implementing the control scheme are limited, such as e.g. when communications between the plant and the controller can only happen through a finite capacity channel. As a consequence of taking such viewpoint, the focal point of research is to understand how to quantize the control system best (in some suitable sense), rather than assessing robustness of design with respect to quantization. In their papers, both Wong and Brockett (1999) and Elia and Mitter (1999) focus on the stabilization problem. Authors of the latter paper provide a result on the optimal (coarsest) quantization for asymptotically stabi-

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lizing a linear discrete–time system, that turns out to require a countable symmetric set of logarithmically decreasing inputs, namely $U = \{\pm u_i : u_{i+1} = \rho u_i, -\infty \leq i \leq +\infty\} \cup \{0\}$. Although this choice (and the corresponding partition induced in the state space) captures the intuitive notion that coarser control is necessary when far from the goal, it still needs input values that are arbitrarily close to each other near the equilibrium.

An observation common to many papers on stabilization with quantized control is that, if the available quantized control set is finite, or countable but nowhere dense (in the natural topology of $\mathbb{R}^m$) then stability can only be achieved in a weak sense — be it ultimate boundedness (Hou *et al.* (1997)), containability (Wong and Brockett (1999)), or practical stability (Elia and Mitter (1999)).

The focus of our paper is on the study of particular phenomena that may appear in quantized control systems, which have no counterpart in classical systems theory, and that deeply influence the qualitative properties and performance of the control system. These concern the structure of the set of points that are reachable by system (1), and particularly its density. We will address two instances of the general system (1), namely linear systems, and driftless nonlinear systems. In particular, among the latter, we will focus our attention on the (discrete counterpart of) nonholonomic systems. We report on conditions under which the rechable set for these systems is dense in $\mathbb{R}^n$, or otherwise when it possesses a lattice structure. We will discuss applications to problems in steering nonholonomic systems, and discuss possible implications and open problems in the analysis and synthesis of quantized control systems.

2. FIRST DEFINITIONS AND EXAMPLES

We will consider systems defined as follows

Definition 1. Let a system be defined by a quintuple $(\mathcal{X}, \mathcal{T}, \mathcal{U}, \Omega, A)$*, where* $\mathcal{X}$ *denotes the configuration set,* $\mathcal{T}$ *an ordered time set,* $\mathcal{U}$ *a set of acceptable input symbols (possibly depending on the configuration),* Ω *a set of acceptable input words, and* $\mathcal{A}$ *is a state–transition map* $\mathcal{A} : \mathcal{T} \times \Omega \times \mathcal{X} \to \mathcal{X}$*. Denote* $\mathcal{A}_{t,\omega}(x) = \mathcal{A}(t, \omega, x)$*, with composition by concatenation* $\mathcal{A}(x_1, a_2, t_1) \circ \mathcal{A}(x_0, a_1, t_0) = \mathcal{A}(\mathcal{A}(\mathbf{x}_0, a_1, t_0), a_2, t_1)$*.*

In particular, we will focus here on $\mathcal{T} = \mathbb{N}$, as most interesting phenomena relating with quantization appear as linked to discrete time. A system as in definition 1 with both $\mathcal{X}$ and $\mathcal{U}$ discrete sets essentially represents a sequential machine or an automaton, while for $\mathcal{X}$ and $\mathcal{U}$ continuous sets, a discrete–time, nonlinear control system is obtained. We are interested in studying reachability problems that arise when $\mathcal{X}$ has the cardinality of a continuum, but $\mathcal{U}$ is discrete (i.e., finite or countable, but nowhere dense), i.e. when inputs are *quantized.* The following example motivates the generality of the definition above with a specific robotics application.

Example1. We will consider the discrete analogue of a well known continuous nonholonomic system, which is the plate–ball system (see e.g. Brockett and Dai (1993); Jurdjevic (1993); Levi (1993)). A ball rolls without slipping between two parallel plates, of which one is fixed and the other one translates. If the moving plate is driven along a closed trajectory, in particular e.g. it is translated to the right by some amount, then forward, left, and backward by the same amount, the same will happen to the ball centre, which will end up in the same initial position. However, the final orientation of the sphere will be changed by a net amount. Indeed, it can be shown (Marigo and Bicchi (2000)) that an arbitary orientation in $SO(3)$ can be reached by rolling arbitrary pairs of non–isomorphic surfaces, which fact was used as a basis for building simplified dextrous robot hands.

Consider now a similar experiment with a polyhedron replacing the ball. For practical reasons, possible actions on this system (studied in detail in Ceccarelli *et al.* (2000)), are only rotations about one of the edges of the face lying on the plate, by exactly the amount that brings an adjacent face on the plate. A generic configuration of the polyhedron can be described by giving the index of the face sitting on the plate, the position of the projection on the plate of the centroid, and the orientation of the projection of an inner diagonal of the cube. Hence, the configuration set is represented by the stratified manifold $\mathcal{X} = \mathbb{R}^2 \times S^1 \times \mathcal{F}$, where $\mathcal{F}$ denotes the set of faces of the polyhedron. Given the discrete nature of input actions, we take $\mathcal{T} = \mathbb{N}_+$, $\mathcal{U} = \mathcal{F}$, Ω the (configuration–dependent) set of all sequences of adjacent faces starting with the face of the present configuration, and $\mathcal{A}_{t,\omega}(x)$ the configuration reached at the end of a t–long sequence of tumbles $\omega \in \Omega$ allowed at x. ◁

Definition 2. A configuration x_f *is reachable from* x_0 *if there exists a time* $t \in \mathcal{T}$ *and an acceptable input string* $\omega \in \Omega$ *that steers the system from* x_0 *to* $x_f = \mathcal{A}_{t,\omega}(x_0)$*.*

In the following we shall denote by R_x the reachable set from x, i.e. the set of configurations that can be reached from x. For differentiable systems, the notion of *reachability from* x is introduced when $R_x = \mathcal{X}$. For discrete–time systems with quantized inputs, however, Ω is a subset of all possible finite sequences ω of symbols in the discrete set $\mathcal{U}$, hence R_x is a countable set and, in the general case that the configuration set $\mathcal{X}$ has the cardinality of a continuum, it will not make sense checking whether R_x equals $\mathcal{X}$.

Example1–b. The set of configurations that can be reached starting from a given configuration of the polyhedron of Example 1, in a large but finite number of steps N, may have different characteristics. Consider for instance (intutively, or by simulation) positions reached by the centroid of different polyhedra after N steps: only points lying on a regular grid can be reached by rolling a cube, while for a generic parallelepiped or pyramid they tend to fill the plane as N grows. Also, orientations obtained by rolling the cube or the parallelepiped are only multiples of $\pi/2$, while orientations reached by the generic pyramid tend to fill the unit circle as N grows. Conditions under which the reachable set is dense, and a description of the lattice structure in discrete cases, have been studied by Y.Chitour *et al.* (1996); Ceccarelli *et al.* (2000). ◁

We introduce a concept of *approachability*, which is stated in the further assumption that the state space is a metric space with distance $d(\mathbf{x}_1, \mathbf{x}_2)$:

Definition 3. A configuration x_f can be approached *from x_0 if $\forall\epsilon$, $\exists t \in \mathcal{T}$, $\exists\omega \in \Omega$ such that $d(\mathcal{A}_{t,\omega}(x_0), x_f) < \epsilon$. We say that the system is* approachable from x_0 *if the reachable set R_{x_0} is dense in $\mathcal{X}$, and is* locally approachable from x_0 *if the closure of the reachable set R_{x_0} contains a neighborhood of x_0. Finally, the system is* approachable *if*

$$\text{closure } (R_{\mathbf{x}}) = \mathcal{X},\ \forall x \in \mathcal{X}.$$

Lack of density of R_x will be referred to as *discreteness* of the reachable set. The term *dense in a subset $\mathcal{X}' \subset \mathcal{X}$* will be used to indicate that

$$\text{closure } (R_x) \cap \mathcal{X}' = \mathcal{X}',\ \forall x \in \mathcal{X},$$

Notice that the possibility that the reachable set of a quantized control system is discrete, separates such systems from differentiable systems; on the other hand, the possibility of having a dense reachable set distinguishes quantized control systems from classical finite–state machines.

In practical aplications, it may be important to measure the coarseness of discrete reachable sets. We will then say that a configuration x_f is ϵ–*approachable* from x_0 if $\exists t \in \mathcal{T}, \omega \in \Omega$, such that $d(\mathcal{A}_{t,\omega}(x_0), x_f) < \epsilon$. The set of configurations that are ϵ–approachable from x is denoted by R_x^ϵ. The system will be said ϵ–approachable if $R_x^\epsilon = \mathcal{X}$, $\forall x \in \mathcal{X}$.

Let us consider a quantized control system in discrete time in the form

$$x^+ = g(x, u),\ u \in U, \tag{2}$$

where $x \in \mathcal{X} = M$, a manifold, and U a finite set. For simplicity, also let Ω be comprised of all strings of symbols in U. Obviously, such definition is equivalent to assigning a finite number of maps $g_u : M \to M$.

In this case the reachable set from a point $x \in M$ is $R_x = \{g_{u_1} \cdots g_{u_n}(x) : n \in \mathbb{N}_0, u_i \in U\}$ ($\mathbb{N}_0$ includes the number 0 so that $x \in R_x$). Moreover, we introduce the relation $\sim$ over the elements of M by setting $x \sim y$, $x, y \in M$, if $y \in R_x$. We want to focus on a special class of systems that we call invertible systems.

Definition 4. The system (2) is said to be invertible if for every $x \in M$ and $u \in U$ there exists a finite sequence of controls $u_i \in U$, $i = 1, \ldots, n$, such that $g_{u_1} \cdots g_{u_n}(g(x, u)) = x$.

The following proposition is obvious:

Proposition 1. The relation $\sim$ is an equivalence relation if and only if the system is invertible.

If the system is invertible, we can partition the state space into a family of reachable sets. This is equivalent to take the quotient $M/\sim$ with respect to the equivalence relation $\sim$. We call the set $\widetilde{M} = M/\sim$ the reachability set of the system (2) and we endow $\widetilde{M}$ with the quotient topology, that is the largest topology such that $\pi : M \to \widetilde{M}$, the canonical projection, is continuous.

Example 2. Consider the system

$$x^+ = x + u$$

where $x \in \mathbb{R}$ and $u \in \mathcal{U}$, $\mathcal{U}$ finite subset of $\mathbb{R}$. If $\mathcal{U} = \{0, 1/2, -1\}$ then the system is invertible. The reachable set from the origin R_0 is the subgroup of $\mathbb{R}$ generated by $1/2$ and the reachability set $\widetilde{M}$ is homeomorphic to S^1. If $\mathcal{U} = \{\sqrt{2}, -1\}$ then the system is not invertible. For example $\sqrt{2} \in R_0$, but, since $\sqrt{2}$ is irrational, $0 \notin R_{\sqrt{2}}$. ◁

Example3. Consider the system

$$x^+ = g(x, u)$$

where $x \in \mathbb{R}$, $\mathcal{U} = \{\pm 1/2, \pm 2\}$ and $g(x, u) = u \cdot x$. The system is invertible, $R_0 = \{0\}$ and for every $x \neq 0$ $R_x = \{\pm 2^i x : i \in \mathbb{Z}\}$. The reachability set $\widetilde{M}$ is homeomorphic to the set $S^1 \cup \{\alpha\}$, where on S^1 there is the usual topology while the only neighborhood of α is the whole space. ◁

Notice that in example 3, the reachable set R_x for $x \neq 0$ has only one accumulation point, namely 0. If we assume that M is a metric space and the maps g_u are isometries then we have a dicotomy illustrated by next proposition:

Proposition 2. Consider an invertible system (2). Let (M, d) be a metric space and assume that $x \to g(x, u)$ is an isometry for every $u \in \mathcal{U}$ then each reachable set R_x is formed either by accumulation points or by isolated points.

Proof. Assume that the set R_x admits an accumulation point $\bar{x} \in R_x$. Let $x_k \in R_x$ be such that $x_k \to \bar{x}$ and the set $\{x_k : k \in \mathbb{Z}\}$ is infinite. Since the system is invertible, for every k there exists $\tilde{u}_k = (u_k^1, \ldots, u_k^{n_k})$ such that $u_k^i \in U$ and $g_{u_k^1} \cdots g_{u_k^{n_k}}(x_k) = x$. Define $y_k = \lim_m g_{u_k^1} \cdots g_{u_k^{n_k}}(x_m)$. For every k and m we have:

$$d(g_{u_k^1} \cdots g_{u_k^{n_k}}(x_m), x) = \\ d(g_{u_k^1} \cdots g_{u_k^{n_k}}(x_m), g_{u_k^1} \cdots g_{u_k^{n_k}}(x_k)) = \\ d(x_m, x_k).$$

Passing to the limit in m, we have $d(y_k, x) = d(\bar{x}, x_k)$. Clearly the sequence y_k converge to x and contains infinitely many distinct points, so x is an accumulation point for R_x. Now it easily follows that all points of R_x are accumulation points for R_x. □

The system:

$$x^+ = x + u \tag{3}$$

with $x \in R^n$ is an interesting special case. It is clear that for every $x_0 \in R^n$ the reachable set $X(x_0)$ from x_0 is equal to $x_0 + X_0$ where X_0 is the reachable set from the origin. The hypothesis of the above Proposition are satisfied. Notice that if $n = 1$ and U is simmetric then the set X_0 is either everywhere dense or nowhere dense in $\mathbb{R}$ (since it is a subgroup of $\mathbb{R}$), hence presenting a stonger dicotomy of the one illustrated by the above Proposition. For $n > 1$ we may have directions along which the reachable set X_0 is dense and directions along which is discrete. This is precisely the case of $n = 2$ and $U = \{(\pm 1, 0), (\pm\sqrt{2}, 0), (0, \pm 1)\}$. Notice that if we define $\pi_v : \mathbb{R}^n \to \mathbb{R}$ to be the ortogonal projection on the direction of the vector v, then $\pi_v(X_0)$ is dense in $\mathbb{R}$ for every v not parallel to $(0, 1)$ (and this corresponds to the fact that the projection of the reachable set is precisely the reachable set of the projection of the system). On the other side, $X_0 \cap \{\lambda v : \lambda \in \mathbb{R}\}$ is discrete for every v not parallel to $(1, 0)$.

Another key aspect of reachability analysis for nonlinear control systems is nonholonomy. Definitions for continuous systems are typically formulated in terms of well–known integrability conditions on the constraint codistribution. For a system such as that in 1, a more general definition is necessary:

Definition 5. A system is said to be nonholonomic at x_0 if it is possible to decompose $\mathcal{X}$ in a projection or base *space $\mathcal{B} = \Pi(\mathcal{X})$ and a* fiber *$\mathcal{F}$, such that $\mathcal{B} \times \mathcal{F} = \mathcal{X}$ (that is $\mathcal{X}$ is a trivial bundle) and there exists $\omega \in \Omega$ and $t \in \mathcal{T}$ such that $\mathcal{A}_{t,\omega}$ steers the system from x_0 to $x^\star = \mathcal{A}_{t,\omega}(x_0)$ with $\Pi(x_0) = \Pi(x^\star)$ but $x_0 \neq x^\star$.*

Example1–c. An illustration of discrete nonholonomy is obtained by the rolling polyhedron system. When a sequence of rotations on the right, hence forward, left and backward is considered, the center of the die returns to its initial position, while the orientation has changed by a finite amount. ◁

3. SYNTHESIS OF LINEAR QUANTIZED CONTROL SYSTEMS

In this section, we report some results on systems of the form

$$x^+ = Ax + Bu, \ u \in U \tag{4}$$

with U a quantized set as usual, and (A, B) a controllable pair. Reachability questions that may be asked about such system can be divided in two types:

Definition 6.

Q1 given a pair (A, B), find conditions under which a quantized control set U exists such that the reachable set $R(0, U)$, from 0 and corresponding to the set U, is dense in $\mathbb{R}^n$. If possible, find such a U.

Q2 given a pair (A, B), a quantized set U, and initial conditions $x(0)$, determine whether or not the corresponding reachable set is dense.

We will refer to question **Q1** as to a synthesis problem, and to **Q2** as to an analysis problem.

The synthesis problem has been extensivley studied in Chitour and Piccoli (2000). Main results are reported below.

Theorem 1. Necessary and sufficient conditions for a quantized control set U to exist such that the reachable set $R(0, U)$ from 0 of (4) is dense in $\mathbb{R}^n$ are that

(1) (A, B) is controllable;
(2) if λ is an eigenvalue of A, then $|\lambda| \geq 1$.

The necessity of the first condition is obvious. If the second condition does not hold, the reachable set is bounded in some component. However, a similar density result can still be obtained (provided that no eigenvalue of A is zero) if local approachability at the origin is considered instead.

Remark. Conditions for a positive answer to the synthesis problem are very weak. Proofs given in Chitour and Piccoli (2000), though far from trivial, are constructive, as they provide explicitly a *standard* control set $\mathcal{U} = \{0, \pm u_1, \pm u_2, \ldots\}$ that achieves density for a fixed system. Furthermore, results are shown to be uniform with respect to both initial conditions and eigenvalues changes.

A further twist to the synthesis problem results from restricting control values to belong to a subset of $\mathbb{R}^n$. In particular, in applications involving D/A conversions, inputs will be restricted as $U \in \mathbb{Q}^n$. The case $U \in \mathbb{Z}^n$ is also relevant to many applications. For this case we immediately have the following:

Theorem 2. Consider the system (4) and assume that the matrices A and B have integer entries. Let $U = \{i\,\alpha : i \in \mathbb{Z}\}$ for some $\alpha \in \mathbb{R}$. Then the reachable set $R(0, U)$ is discrete.

In general if we allow the control set U to be discrete but infinite then unless we are in the situation of the above theorem we expect density of $R(0, U)$ to be generic. The situation is profoundly different if we consider finite control set U even without uniform bound on the cardinality. There is a special class of algebraic numbers that play a key role. We recall that an algebraic number λ is a real number that is root of a polynomial P with integer coefficients. If, moreover, the leading coefficient of P is 1 then λ is called an algebraic integer. For an algebraic number λ we can determine the minimal polynomial P_λ that is the polynomial of minimal degree such that $P_\lambda(\lambda) = 0$, moreover if λ is an algebraic integer P_λ can be chosen with leading coefficient 1. Given an algebraic number λ we call the other roots of P_λ the Galois conjugates of λ (obviousy they cannot be real).

Definition 7. An algebraic integer $\lambda > 1$ is a Pisot number if all its Galois conjugates have modulus strictly less than one.

The following theorem follows form the analysis of Chitour and Piccoli (2000).

Theorem 3. Consider a system (4) satisfying the assumptions of Theorem 1 (necessary for density) and assume that A is in Jordan form, $B = I$ (the identity matrix). The reachable set $R(0, U)$ is not dense in $\mathbb{R}^n$ for every finite set $U \subset \mathbb{Q}^n$ if and only if there exists an eigenvalue of A whose modulus is a Pisot number.

Notice the strength of the Theorem implying that in the case in which an eigenvalue (or its modulus) is a Pisot number, then whatever choice of a finite set $U \subset \mathbb{Q}^n$ with arbitrarily large finite cardinality gives a nondense reachable set $R(0, U)$. The set of Pisot number is obviously countable but the surprising fact is that it is close. Hence, it is not dense in $\mathbb{R}$ and indeed is "small" in topological sense. Many facts are indeed known about the set T of Pisot numbers. For example T admits a minimum value $\lambda \sim 1.33$, that is the unique positive root of $x^3 - x - 1$. The smallest accumulation point of T is the well known golden number $(1 + \sqrt{5})/2$ that is root of $x^2 - x - 1$. We refer the reader to Chitour and Piccoli (2000) and references therein for information about Pisot numbers.

On the other side, if all eigenvalues are not Pisot then it is possible to obtain density of $R(0, U)$ choosing a large enough number M (of the order of the modulus of the biggest eigenvalue) and all controls with integer coordinates in $[-M, M]$. See Erdös *et al.* (1998) and Chitour and Piccoli (n.d.).

We want also to point out that sampled systems with D/A conversions and usage of computers naturally lead to system of type (4) with U finite subset or $\mathbb{Q}^n$. It is then clear the importance of the above result.

4. ANALYSIS PROBLEMS

The analysis question is indeed much more difficult to answer. To understand the difficulty we refer the reader to Keane *et al.* (1995) where the so called $\{0, 1, 3\}$-problem is studied. This corresponds exactly to the analysis of the Hausdorff measure of the reachable set for the system $x^+ = \lambda x + u$, $x \in \mathbb{R}$, $\lambda < 1$, $u \in U = \{0, 1, 3\}$, if we allow infinite sequences of controls. The analysis problem has some partial answer in the cited paper and references therein.

Another strictly linked number theory problem is the one considered in Erdös *et al.* (1998). We refer the reader to Chitour and Piccoli (2000) for a deeper discussion of the links between these hard mathematical problems. From the results of Erdös *et al.* (1998) it is even more clear the role played by Pisot numbers.

In this section, we provide some results on the analysis question concerning some simple examples of driftless systems of the type

$$x^+ = x + g(x)u, \; u \in U \tag{5}$$

Given two real numbers $r_1, r_2 \in \mathbb{R}$ we write $r_1 \sim r_2$ to indicate that r_1, r_2 have rational ratio, that is $\frac{r_1}{r_2} \in \mathbb{Q}$. It is easy to check that $\sim$ is an equivalence relation. Consider the control system

$$x^+ = x + u \tag{6}$$

where $x \in \mathbb{R}$ and u takes values in a finite set $U \subset \mathbb{R}$. Our aim is to prove that the following condition is necessary and sufficient in order to have that the reachable set from any initial point is dense in $\mathbb{R}$.

(C) There exist $u, v \in U$ such that $u \not\sim v$ and $u \cdot v < 0$.

First notice that condition (C) is equivalent to the following

(C') There exist $u, v \in U$ such that $u \not\sim v$ and there exist $u', v' \in U$ such that $u' \cdot v' < 0$.

Indeed, obviously (C) implies (C'). On the other hand, assume that (C') is true, then $U^{\pm} = U \cap R^{\pm}$ are nonempty. If for every $u \in U^+$ and $v \in U^-$ we have $u \sim v$ then, since $\sim$ is an equivalence relation we get that all control have rational ratio reaching a contradiction.

We start noticing the following fact (see e.g. Chitour and Piccoli (2000)):

Proposition 3. Let R_0 be a reachable set for the system (6) from the origin. Then R_0 is dense if and only if there exist two sequences $c_k \in R_0$ and $d_k \in R_0$ both converging to zero such that $d_k < 0 < c_k$.

Let us now prove the following

Theorem 4. Let R_0 be a reachable set for the system (6) from the origin. Then R_0 is dense if and only if (C) holds true. Moreover, if R_0 is not dense then is nowhere dense.

Proof. Let us first assume that (C) holds true and let $u, v \in \mathcal{U}$ be as in (C). Since the ratio $\frac{u}{v}$ is not rational we can consider the sequence $\frac{p_k}{q_k} \in Q$, p_k, q_k integers, $q_k > 0$, given by its continued fraction. We have:

$$\frac{u}{v} - \frac{p_k}{q_k} = (-1)^k \varepsilon_k$$

where $0 < \varepsilon_k < \frac{1}{q_k^2}$ and q_k grows to infinity. We get immediately:

$$q_k \, u + (-p_k)v = (-1)^k v \varepsilon_k q_k.$$

From $u \cdot v < 0$ we get $-p_k > 0$, hence $q_k \, u + (-p_k)v \in R_0$. Now the required sequences are obtained setting, if $v > 0$, $c_k = q_k \, u + (-p_k)v$ for k even and $d_k = q_k \, u + (-p_k)v$ for k odd and the opposite if $v < 0$.

Assume now that (C) does not hold. Then either $u \cdot v > 0$ for every $u, v \in \mathcal{U}$ or $u \sim v$ for every $u, v \in \mathcal{U}$. In the first case it is obvious that the set R_0 is contained either in R^+ or in R^-. In the latter case, the proof is as follows. Let $\mathcal{U} = \{u_1, \ldots, u_n\}$ with $u_1 \neq 0$. Then any point of the reachable set R_{x_0} from x_0 can be written as $x_0 + a$, $a = m_1 \, u_1 + \ldots + m_n \, u_n$ with u_i positive integers. We have $\frac{u_i}{u_1} = \frac{p_i}{q_i} \in Q$ ($\frac{p_1}{q_1} = 1$), thus:

$$a = m_1 \, u_1 + \ldots + m_n \, u_n = u_1 \left(\sum_{i=1}^{n} \frac{m_i \, p_i}{q_i} \right) =$$

$$= u_1 \left(\frac{\sum_{i=1}^{n} m_i \, p_i \, q_1 \cdots q_{i-1} q_{i+1} \cdots q_n}{q_1 \cdots q_n} \right).$$

Now if $a \neq 0$ we have that the numerator of the above expression is different from zero and being an integer is at least of modulus 1. Therefore, if $a \neq 0$ we get

$$|a| > \frac{|u_1|}{|q_1 \cdots q_n|}$$

and obviously R_0 can not be dense. Moreover, from the same expression we have that a is always a multiple of $u_1 / (q_1 \cdots q_n)$ hence R_0 is indeed nowhere dense. □

Since the reachable set from a point x_0 is exactly $x_0 + R_0$ we have a dicotomy similar to that of Section 2, even if in this case (due to the possible non simmetry of $\mathcal{U}$) R_0 may fail to be a subgroup of $\mathbb{R}$.

Let us consider the system (6) but now with $x \in \mathbb{R}^n$, that is

$$x^+ = x + u \tag{7}$$

with $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^n$. From the above analysis we get:

Theorem 5. A necessary condition for the reachable set X from the origin to be dense is that $\mathcal{U}$ contains $n+1$ controls of which n are linealy independent. If $u_1, \ldots, u_n \in \mathcal{U}$ are linearly independent and there exists n irrational negative numbers $\alpha_1, \ldots, \alpha_n$ such that $v_i = \alpha_i u_i \in \mathcal{U}$ for every $i = 1, \ldots, n$ then X is dense in $\mathbb{R}^n$.

5. NONHOLONOMIC SYSTEMS

We are interested in studying the structure of the reachability set for nonlinear system that exhibit nonholonomic behaviours. To do so, we consider the discrete–time analog of a much studied class of continuous–time nonholonomic systems that are written in chained form

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
\vdots &= \vdots \\
\dot{x}_n &= x_{n-1} u_1
\end{aligned}
\quad . \qquad (8)
$$

The chained form was introduced in R. M. Murray (1993) because it allows a rather simple steering method, using sinusoids at integrally related frquencies. The technique consisted in driving system (8) to the desired value of the variables x_1, x_2; then applying a low frequency sinusoidal input to steer x_3 while bringing back x_1, x_2 after a cycle; and so on iteratively with higher frequency sinusoids. At each step, the amplitude of the sinusoids is adjusted so as to reach the desired value of the corresponding coordinate.

A different technique for steering continuous nonholonomic systems that are in strictly triangular form [2] has been proposed in Marigo and Bicchi (1998). The idea there was to purposefully introduce quantization of the input space, by defining a set of fixed input functions on compact time sets. Such control "quanta" can then be concatenated, and form a group acting on the left on the configuration space. The ST form of the system guarantees that the action of the subgroup of the control quanta group that takes the base variables (x_1, x_2) back to their initial value, is abelian on variables x_3. Furthermore, the action of proper subgroups (the derived flag of the control quanta group) is also abelian on corresponding sections of the fiber. Although an infinite number of generators for the control quanta group should in principle be considered, authors proposed to use a finite set generating a discrete reachable set with a lattice structure. These properties allow to steer the system to a desired configuration variable after variable, by simply writing the lattice generators in Hermite normal form, planning on the lattice, then using the generalized inverse Euclid algorithm.

Consider now the discrete system

$$
\begin{aligned}
x_1^+ &= x_1 + u_1 \\
x_2^+ &= x_2 + u_2 \\
x_3^+ &= x_3 + x_2 u_1 + u_1 u_2 \frac{1}{2} \\
x_4^+ &= x_4 + x_3 u_1 + x_2 u_1^2/2 + u_1^2 u_2 \frac{1}{6} \\
\vdots &= \vdots \\
x_n^+ &= \sum_{i=0}^{n-2} x_{n-i} \frac{u_1^i}{i!} + u_1^{n-2} u_2 \frac{1}{(n-1)!}
\end{aligned}
\qquad (9)
$$

[2] A system is in ST form if $\dot{x}_i = g(x_{i+1}, \cdots, x_n)u$. ST systems include, but are not limited to, nilpotent systems Marigo (1999), and are hence much more general than chained form systems.

which can be regarded as system (8) under unit sampling. Notice that this system is invertible (as opposed e.g. to the forward Euler approximation of (8)). Indeed, for any state–independent, symmetric set of input symbols $\mathcal{U}$, the group of input words $\Omega = \{\text{strings of symbols in } \mathcal{U}\}$ with inverse $(w_1 w_2 \cdots w_m)^{-1} = -w_m \cdots - w_b - w_a$, $\pm w_i \in \mathcal{U}, \forall i$, acts on the configuration space through the end–point map such that $\mathcal{A}(\omega^{-1}, \mathcal{A}(\omega, x)) = x$.

We are interested in studying the reachability set of system (9), and in providing a steering method for the system.

One can readily check that the system is nonholonomic in the sense of definition 5, by taking (x_1, x_2) as the base variables. Reachability in the base space can be studied by results reported above for linear driftless systems. We will hence focus on the reachability of the fiber corresponding to a given base point $(\bar{x}_1, \bar{x}_2)$. Simple calculations show that the reachable set in the fiber does not depend on the base variables, hence we may consider $\bar{x}_1 = 0, \bar{x}_2 = 0$ without loss of generality.

Consider the subgroup $\tilde{\Omega} \in \Omega$ of control words that take the base variables back to their initial configuration. These are sequences of inputs such that the sum of the first and second components are zero. The action of this subgroup on the fiber is commutative: namely, $\mathcal{A}(\tilde{\omega}_1, \mathcal{A}(\tilde{\omega}_2, x)) = \mathcal{A}(\tilde{\omega}_2, \mathcal{A}(\tilde{\omega}_1, x))$, $\forall \tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{\Omega}$. Notice that this represents a significant departure from the behaviour of the continuous model (8), where the action of the generic cyclic control is abelian only on the first fiber variable, x_3, and more restricted subgroups should be searched that have the commutative action property on the rest of the fiber.

To be more specific, let us consider the case that $\mathcal{U} = \{\pm(1,0), \pm(0,1), \pm(a,b)\}$. The rechable set for the base variables is described by

$$
\begin{aligned}
x_1 &= m_1 + a m_3 \\
x_2 &= m_2 + b m_3
\end{aligned}
\qquad (10)
$$

with $m_i \in \mathbb{Z}$. If a and b are rational, and $a = \frac{p_a}{q_a}$, $b = \frac{p_b}{q_b}$, p_a, q_a, p_b, q_b integers and pairwise coprime, the rechable set of base space is clearly a lattice. The subgroup $\tilde{\Omega}$ is given by all control words with $(m_1, m_2, m_3) = \alpha(p_a q_b, p_b q_a, -q_a q_b)$, $\alpha \in \mathbb{Z}$ (this means, words where the symbol $(1,0)$ is used mm_1 times, $(0,1)$ is used m_2 times, and $(-a,-b)$ is used m_3 times). For $\alpha = 1$, there are

$$
N = \frac{(p_a q_b + p_b q_a + q_a q_b)!}{p_a q_b! p_b q_a! q_a q_b!}
$$

possible words. The reachable set as a whole is discrete: the i–th coordinate is an integer multiple of $1/\mu_i$ dove $\mu_i = (i-1)! q^(i-1)$, $q = \max\{q_a, q_b\}$. Coordinates of higher index have a finer resolution.

6. CONCLUSIONS

In this paper, we have considered reachability problems in quantized control systems. We have shown that the reachable set may be dense or discrete depending on the quantized set of inputs, and have provided some results in the analysis and synthesis problems. We have also provided a definition and some characterization of nonholonomic phenomena occurring in nonlinear quantized control systems. Many open problems remain in this field, that is in our opinion among the most important and challenging for applications of embedded control systems and in several other applications. Although some problems have been shown to hard, we believe that a reasonably complete and useful system theory of quantized control system could be built by merging modern discrete mathematics techniques with classical tools of system theory.

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