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A Group-Theoretic Characterization of Quantized Control Systems

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Abstract

In this paper we consider the reachability problem for quantized control systems, i.e. systems that take inputs from a finite set of symbols. Previous work addressed this problem for linear systems and for some specific classes of nonlinear driftless systems. In this paper we attack the study of more general nonlinear systems. To do so we find it useful to pose the problem in more abstract terms, and make use of the wealth of tools available in group theory, which enables us to proceed in our agenda of better understanding effects of quantization of inputs on dynamic systems.

1 Introduction

Quantized control systems often represent a proper model to deal with several real-world control systems, among which for instance are applications using switching actuators, qualitative measurements, or plants where the hardware implementation of the controller loop only admits information transfer with a finite bandwidth.

Several seminal contributions have appeared in recent years on such problems, including those of [3, 6, 7, 8]. In our previous work we have considered in some detail the analysis of the reachable set and the synthesis of open-loop controls. A typical question arising under this regard is whether, for a given set of input symbols, the recahble set is everywhere dense or not, and if not, if there are useful structures in the reachable set, such as e.g. a lattice structure.

These questions, which have a direct bearing on steering systems from one state to another and indirectly also affect stabilization policies, have been answered in [2, 9] for a particular class of systems, i.e. nonlinear driftless systems in chained form. Although this class is rather broad and interesting in applications (most nonholonomic systems can be written in such form by a suitable feedback and state diffeomorphism), in this paper we aim at generalizing the approach. In particular, we will focus here on nonlinear driftless systems which are not in chained form, and are subject to quantized inputs. Two examples will be considered for illustration: the case of a rolling polyhedron (which is the quantized counterpart of the plate-ball system, hence is not equivalent to chained form), and the n-trailer vehicle system (for which a feedback transformation to chained form only exists if the control can take continuous values). Our program is to embed these more general problems into the general framework of group actions so as to reduce the basic questions of density/discreteness of reachable sets to the study of normal subgroups, for which a wealth of tools are available form group theory.

The action of sequences of controls can be formalized, under suitable assumptions, as a group action of a set of words. Invertibility of control action is required. In general the set of controls depends on the state and we first stratify the state space by equivalence of control sets. Then we focus on the action of the group on a single equivalence class considering words for which the equivalence class is invariant. Most of literature in group action theory is dedicated to the case of Lie groups, but in our case the discreteness of control sets force us to remain at level of general groups. Orbits for the group action are precisely the reachable sets for the system. We introduce additional assumptions to have homogeneity of the space of orbits and show that, if isotropy groups coincide along an orbit, then, up to a quotient, we can reduce to a free action.

Then we introduce our main tool: normal subgroups. In general, given a normal subgroup H, the action can be split in two parts: first the action of the subgroup H on its (sub)orbit and then the action of the quotient over H on the set of H orbits.

This splitting can be viewed as a base-fiber splitting of the state space and it is natural to describe nonholonomic behavior. For the polyhedron example (as well as for isometry groups over $\mathbb{R}^n$) the set of translations, obtained by rotation along edges, is a normal subgroup and the corresponding fibration was used in [4] to detect density of reachable sets. Another important example is that of chain systems in sampled integrated form, see [2, 9]. Also in this case a complete classification of topologies of reachable sets was obtained through a natural base-fiber reduction.

Action of isometries is of particular interest, since in this case we have that the reachable set is formed either by accumulation points or by isolated ones. However, we possibly expect that only the action of the elements

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of some subgroup H are isometries on the relative orbits and, in particular, that such action is linear. In this case the work of [5] may help to understand the structure of the reachable set. Moreover, for the natural action of the group modulo H over the space of H orbits, we introduce the pseudometric that collapses to zero distance all elements of each H orbit. This approach works even if the whole set of words fails to be a group, as e.g. for the n–trailer system.

2 Definitions and Fundamental Assumptions

We begin with describing a quantized control system in the language of the theory of groups. A discrete time–invariant quantized control system is a 4–tuple $(\mathcal{Q}, \mathcal{U}, \mathcal{A}, \Omega)$ with $\mathcal{Q}$ denoting the configuration set, $\mathcal{U}$ a set of admissible input symbols, $\mathcal{A}$ a state–transition map $\mathcal{A} : \mathcal{Q} \times \mathcal{U} \to \mathcal{Q}$. Notice that in general $\mathcal{U}$ is to be considered as state-dependent, hence $\mathcal{A}(q, u)$ is well defined if $u \in \mathcal{U}$ is admissible for $q \in \mathcal{Q}$. Moreover if u_1 is admissible for q and u_2 is admissible for $\mathcal{A}(q, u_1)$ then we say that u_1 and u_2 are concatenable from q and denote $u_1 u_2$ the concatenation of u_1 and u_2. By recursion, we define an "admissible input stream" from a point $q \in \mathcal{U}$ to be the concatenation of concatenable symbols in $\mathcal{U}$ from $q \in \mathcal{Q}$, denote by Ω_q the set of admissible input streams from q and set $\Omega = \cup_{q \in \mathcal{Q}} \Omega_q$[1]. Next we give more structure to our sets in order to have a suitable representation for the transition map $\mathcal{A}$.

Consider the multivalued function $\phi : \mathcal{Q} \to \mathcal{U}$ where $\phi(q) = \mathcal{U}_q \subset \mathcal{U}$ is the set of admissible inputs at q. Consider the equivalence relation on $\mathcal{Q}$ given by $q_1 \sim q_2$ iff $\phi(q_1) = \phi(q_2)$, denote by $\mathcal{Q}/\phi$ the set of equivalence classes and by $[q]$ the equivalence class of q. We assume the following

(H0) Each equivalence class is a connected submanifold of $\mathcal{Q}$.

Thus we have that the map $\mathcal{A}$ is well defined on each of the product $[q_1] \times \mathcal{U}_{q_1}$:

$$\mathcal{A} : [q_1] \times \mathcal{U}_{q_1} \to \mathcal{Q}$$

where $\mathcal{A}(q_2, u)$ is the state that the system reaches from $q_2 \in [q_1]$ under $u \in \mathcal{U}_{q_1}$. Notice that, in general $\mathcal{A}(q_2, u) \not\sim q_2$. Consider the following condition:

(H1) $\forall q_1 \sim q_2$ **and** $\forall u \in \mathcal{U}_{q_1} (= \mathcal{U}_{q_2})$, $\mathcal{A}(q_1, u) \sim \mathcal{A}(q_2, u)$.

Condition **(H1)** is referred to as the compatibility of the map $\mathcal{A}$ with respect to the equivalence relation $\sim$. It implies the following:

[1] In general we may have $\Omega \subset \cup_{q \in \mathcal{Q}} \Omega_q$ with strict inclusion. For sake of simplicity we restrict ourselves to the case of equality.

Proposition 1 *Assume condition* **(H1)** *holds. If* $q_1 \sim q_2$ *then* $\Omega_{q_1} = \Omega_{q_2}$ *and we can define the map, which, by slight abuse of notation, is also denoted by* $\mathcal{A}$, $\mathcal{A} : [q_1] \times \Omega_{q_1} \to \mathcal{Q}$ *setting*

$$\mathcal{A}(q_2, \omega) = \mathcal{A}(\mathcal{A}(\cdots \mathcal{A}(q_2, u_1), \cdots, u_{N-1}), u_N),$$

the state that the system reaches from $q_2 \in [q_1]$ *under* $\omega = u_1 \cdots u_N \in \Omega_{q_2} (= \Omega_{q_1})$. *Moreover* $\mathcal{A}(q_1, \omega) \sim \mathcal{A}(q_2, \omega)$, *that is the new map* $\mathcal{A}$ *is compatible with the equivalence relation* $\sim$.

Denote by $\widetilde{\Omega}_q = \{\omega \in \Omega_q : \mathcal{A}(q, \omega) \in [q]\}$ the subset of input streams steering the system back to the same equivalence class of the initial point. By **(H1)** $\forall q_1 \sim q_2$ we have $\widetilde{\Omega}_{q_1} = \widetilde{\Omega}_{q_2}$.
Notice that the set Ω with the operation of concatenation is a monoid with neutral element e, the empty word. Let $\mathcal{S}$ be the set of mappings $\mathcal{Q}^{\mathcal{Q}}$ of $\mathcal{Q}$ into itself. Then, the map $a : \Omega \to \mathcal{S}$, with $a(\omega) = \mathcal{A}_\omega$ and $\mathcal{A}_\omega(q) = \mathcal{A}(q, \omega)$, is a monoid homomorphism if we endow $\mathcal{S}$ with the composition operation and define $a(e) = \mathcal{A}_e = Id$, where Id is the identity map of $\mathcal{Q}$. Let $\widetilde{\mathcal{S}} \subset \mathcal{S}$ be the subset of bijective, hence invertible, maps of $\mathcal{Q}$ into itself. Then $\widetilde{\mathcal{S}}$ is a group for the composition operation. To endow $\widetilde{\Omega}_q$ with a group structure we assume the following condition:

(H2) $\forall q \in \mathcal{Q}$, $a(\widetilde{\Omega}_q) \subset \widetilde{\mathcal{S}}$ **and** $\forall \omega \in \widetilde{\Omega}_q$ **there exists** $\bar{\omega} \in \widetilde{\Omega}_q$ **such that** $\mathcal{A}_{\bar{\omega}} = (\mathcal{A}_\omega)^{-1}$.

Remark. *Observe that discrete time–invariant quantized control systems obtained by exact sampling of continuous–time driftless systems satisfy condition* **(H2)**, *provided that the set of input symbols* $\mathcal{U}$ *is symmetric, i.e.* $u \in \mathcal{U} \Rightarrow -u \in \mathcal{U}$
Consider the set of relations $\omega\bar{\omega} = \bar{\omega}\omega = e$, then the quotient of $\widetilde{\Omega}_q$, over the corresponding equivalence relation, is a group. For notational simplicity we still denote by $\widetilde{\Omega}_q$ its quotient. We have that: $a : \widetilde{\Omega}_q \to \widetilde{\mathcal{S}}$ is a group homomorphism, this means $\mathcal{A}_{\omega_1 \omega_2} = a(\omega_1 \omega_2) = a(\omega_2) a(\omega_1) = \mathcal{A}_{\omega_2} \mathcal{A}_{\omega_1}$. Moreover, by **(H2)**, $\mathcal{A}(\mathcal{A}(q, \omega), \bar{\omega}) = \mathcal{A}(q, \omega\bar{\omega}) = \mathcal{A}(q, e) = q$. Finally, for all equivalence classes, we have an action of the group $\widetilde{\Omega}_q$ on $[q]$ with transition map $\mathcal{A}$.

(H3) For all pairs $[q_1], [q_2] \in \mathcal{Q}/\phi$ **there exists** $\omega \in \Omega_{q_1}$ **such that** $\mathcal{A}(\cdot, \omega) : [q_1] \to [q_2]$ **is an homeomorphism and the map** $h : \widetilde{\Omega}_{q_1} \to \widetilde{\Omega}_{q_2}$**, given by** $\omega_1 \in \widetilde{\Omega}_{q_1} \mapsto \bar{\omega}\omega_1\omega = \omega_2 \in \widetilde{\Omega}_{q_2}$**, is a group isomorphism.**

Condition **(H3)** implies that the groups $\widetilde{\Omega}_q$ are conjugate and the map $\mathcal{A}(\cdot, \omega)$ is a h–homeomorphism.
Remark. *This means that we can study the action of one of the groups* $\widetilde{\Omega}_q$ *on the equivalence class* $[q]$ *because for the other equivalence classes we have the same behavior of the action.*

Example 1. Consider a polyhedron rolling on a plane around the edges. The configuration of the polyhedron is determined assigning the face that lies on the plane, the position, and orientation of this face with respect to a coordinate system on the plane. Thus the state space is given by $\mathcal{Q} = \mathcal{F} \times \mathbb{R}^2 \times S^1$, where $\mathcal{F} = \{F_1, \ldots, F_n\}$ is the set of faces of the polyhedron.
Fix $q = (F_i, \bar{x}, \bar{\theta})$, then the possible controls are determined by the edges of the face F_i. Indeed the possible actions are rotations around one of such edges until a face of the polyhedron adjacent to F_i lyes on the plane. Therefore, if we denote by $\{F_j : j \in J_i\}$, $J_i \subset \{1, \ldots, n\}$, all adjacent faces to F_i, we can describe the set of inputs admissible at q as $\mathcal{U}_q = \{F_j : j \in J_i\}$. Then Ω_q is the set of words $F_{i_1} \cdots F_{i_m}$, $m \in \mathbb{N}$, such that $i_1 \in J_i$ and $i_j \in J_{i_{j-1}}$, $j = 2, \ldots, m$.
Each equivalence class $[q]$, $q = (F_i, \bar{x}, \bar{\theta})$, is given by $\{(F_i, x, \theta) : x \in \mathbb{R}^2, \theta \in S^1\}$. Assumptions **(H0)-(H1)** are obviously verified. Notice that $\widetilde{\Omega}_q$ is formed by the words $F_{i_1} \cdots F_{i_m} \in \Omega_q$ such that $i_m = i$. Since every action F_j, $j \in J_i$, is invertible, we get that **(H2)** is also verified.
Now, given two equivalence classes $[q_1] = (F_{i_1}, \cdot, \cdot)$ and $[q_2] = (F_{i_2}, \cdot, \cdot)$, let ω be any word steering the polyhedron from $[q_1]$ to $[q_2]$. Then the map $\mathcal{A}(\cdot, \omega)$ is clearly an homeomorphism of $\mathbb{R}^2 \times S^1$. Moreover the corresponding map h is a group isomorphism so **(H3)** holds true. We thus can fix some q and study the action of the group $\Omega = \widetilde{\Omega}_q$ on $\mathcal{Q} = [q] \simeq \mathbb{R}^2 \times S^1$. From previous works ([4]) we have that $a(\Omega)$ is a subgroup of the group of rigid motions of polygons on the plane $(\mathcal{A}_\omega(\bar{x}, \bar{\theta}) \mapsto (\bar{x} + R(\bar{\theta})t, \bar{\theta} + \psi)$ where $\psi \in S^1$, $t \in \mathbb{R}^2$, depend on ω and $R(\bar{\theta})$ is the matrix of plane rotation of angle $\bar{\theta}$).

From now on we then assume conditions **(H0)**,...,**(H3)** and restrict ourselves to an action of a group Ω on the connected manifold $\mathcal{Q}$:

$$\mathcal{A} : \mathcal{Q} \times \Omega \to \mathcal{Q}.$$

We are interested in the analysis of the reachable set of a quantized control system. In our framework it means that we analyse, from a topological and measure point of view, the orbit of $a(\Omega)$ from a point $q \in \mathcal{Q}$ that is $\mathcal{R}_q = \{\mathcal{A}_\omega(q) : \omega \in \Omega\}$. The set of orbits is given by the quotient $\mathcal{Q}/a(\Omega)$.
We say that an action is transitive if $\forall q_1, q_2 \in \mathcal{Q}$ there exists $\omega \in \Omega$ such that $q_2 = \mathcal{A}(q_1, \omega)$. Since Ω is a discrete group we never have a transitive action. Clearly the action is always transitive on one orbit $\mathcal{R}_q$ and we say that $\mathcal{R}_q$ is a homogeneous Ω–set.
By definition, the action of $\mathcal{A}$ is effective if $\ker a = \{e\}$. If the action is effective we have that $\forall \omega \in \Omega, \omega \neq e$, there exists $q \in \mathcal{Q}$, such that $\mathcal{A}(q, \omega) \neq q$. If $\ker a = N$ then we have that the action of Ω/N on $\mathcal{Q}$ is effective hence, up to quotient, we can assume that we have an effective action of Ω on $\mathcal{Q}$.

With this assumption we then have that, if $\mathcal{A}(\mathcal{A}(q, \omega_1), \omega_2) = q$ for all q, then $\omega_1\omega_2$ is identified with the identity element e, hence ω_1 is identified with $\bar{\omega}_2$. Observe that, even if the action is effective we can have fixed points, i.e. points $q \in \mathcal{Q}$ such that $\mathcal{A}(q, \omega) = q$ for all $\omega \in \Omega$. We denote by Ω^q, $\Omega^q = \{\omega \in \Omega : \mathcal{A}_\omega(q) = q\}$, the isotropy group for q, that is the subgroup of Ω which fixes the point q. We say that the action is free if $\Omega^q = \{e\}$, $\forall q \in \mathcal{Q}$.

(H4) $\forall q_1, q_2 \in \mathcal{Q}$ **with** $q_2 \in \mathcal{R}_{q_1}$ **we have** $\Omega^{q_2} = \Omega^{q_1}$.

Proposition 2 *If* **(H4)** *holds then for every $q \in \mathcal{Q}$, Ω^q is a normal subgroup of Ω and Ω/Ω^q acts freely and transitively on the orbit $\mathcal{R}_q$. We say that $\mathcal{R}_q$ is a homogeneous principal Ω/Ω^q–set.*

Proof: Fix $q_1 \in \mathcal{Q}$, $\tilde{\omega} \in \Omega^{q_1}$ and $\omega \in \Omega$. We need to show that $\omega\tilde{\omega}\bar{\omega} \in \Omega^{q_1}$. Let $q_2 = \mathcal{A}_{\bar{\omega}}(q_1) \in \mathcal{R}_{q_1}$, then

$$\begin{aligned} &\mathcal{A}_{\omega\tilde{\omega}\bar{\omega}}(q_2) = \mathcal{A}_{\omega\tilde{\omega}\bar{\omega}}\left(\mathcal{A}_{\bar{\omega}}(q_1)\right) = \\ &\mathcal{A}_{\bar{\omega}\omega\tilde{\omega}\bar{\omega}}(q_1) = \mathcal{A}_{\tilde{\omega}\bar{\omega}}(q_1) = \\ &\mathcal{A}_{\bar{\omega}}\left(\mathcal{A}_{\tilde{\omega}}(q_1)\right) = \mathcal{A}_{\bar{\omega}}(q_1) = q_2, \end{aligned}$$

hence $\omega\tilde{\omega}\bar{\omega} \in \Omega^{q_2} = \Omega^{q_1}$. ∎

Thus if **(H4)** holds, then, up to quotient, it is not restrictive to assume that $\Omega^q = \{e\}$, hence that $\mathcal{R}_q$ is a homogeneous principal Ω–set. If we have more than one orbit we would like the structure of different orbits to be always the same, from a qualitative point of view. This is guaranteed if we assume

(H5) For all $q_1, q_2 \in \mathcal{Q}$ **there exists a homeomorphism** $\varphi : \mathcal{Q} \to \mathcal{Q}$, $\varphi(q_1) = q_2$ **such that for every** $\omega \in \Omega$ **we have** $\varphi\mathcal{A}_\omega = \mathcal{A}_\omega\varphi$.

If **(H5)** holds we get that φ establishes a bijection between $\mathcal{R}_{q_1}$ and $\mathcal{R}_{q_2}$. Moreover the two reachable sets have the same topological properties.
From now on we also assume **(H4)-(H5)**, fix one point $\bar{q} \in \mathcal{Q}$ and restrict ourselves to the analysis of the orbit $\mathcal{R}_{\bar{q}}$.

Example 1.(continued) In the polyhedron example we have that the isotropy group $\Omega^q = \ker(a)$, because if ω fixes a point then it fixes all points, hence $\omega \in \ker(a)$. Therefore assumption **(H4)** is verified. We consider the action of $\Omega/\ker(a)$ on the orbits $\mathcal{R}_q$, that is free and transitive. We also have that the orbits are isometric. Indeed consider q_1, q_2 any two points of $\mathcal{Q}$ and φ a rotation of the S^1 component followed by a translation of the $\mathbb{R}^2$ component such that $\varphi(q_1) = q_2$. Then φ is an isometry and satisfies **(H5)**. We thus can restrict our study to a single orbit.
A simple example is given by the manipulation of a cube with side of length ℓ. Fix a face, say F_1, and consider the orbit through $(x, \theta) \in \mathbb{R}^2 \times S^1$. Then we

can reach all points with first component on a square lattice of side ℓ and orientation of type $\theta + k\pi/2$.

3 Subgroup actions and base–fiber decompositions

Let $H \subset \Omega$ be a subgroup. Since a is a group homomorphism, then $a(H) \subset a(\Omega)$ is a subgroup ($\omega_1, \omega_2 \in H \Rightarrow \mathcal{A}_{\omega_1}, \mathcal{A}_{\omega_2} \in a(H)$ and $\mathcal{A}_{\omega_1}\mathcal{A}_{\omega_2} = \mathcal{A}_{\omega_2\omega_1} \in a(H)$). Therefore, we can consider the orbit of q under the action of H and, denoting it by $\mathcal{R}_q^H$, we clearly have $\mathcal{R}_q^H \subset \mathcal{R}_q$. In particular we notice that Ω^q, the isotropy group of q, is a subgroup of Ω and $\mathcal{R}_q^{\Omega^q} = \{q\}$ (this holds true even if $\Omega^q \neq \{e\}$).

If H is a normal subgroup of Ω then, by definition, $\forall \omega \in \Omega$, $\omega H \bar{\omega} = H$ and Ω/H is a group. As $\mathcal{R}_q$ is a homogeneous principal Ω–set, $\mathcal{R}_q^H$ is also a homogeneous principal H–set. This approach allowes us to first study the action of the normal subgroup H on q and, second, the action of Ω/H on the set of orbits $\{\mathcal{R}_{q'}^H : q' \in \mathcal{R}_q\}$.

We observe that if $q_2 = \mathcal{A}_h(q_1)$, with $h \in H$, then, by definition of normal subgroup, there exists $h' \in H$ such that

$$\mathcal{A}_\omega(q_2) = \mathcal{A}_{h\omega}(q_1) = \mathcal{A}_{\omega h'}(q_1).$$

Then

$$\mathcal{A}_\omega(q_2) = \mathcal{A}_{h'}\mathcal{A}_\omega(q_1),$$

i.e. two different points of the same H–orbit are mapped by any element of Ω to the same H–orbit. But, since h is in general different from h', the operation of moving along the H–orbit does not commute with that of moving through different H–orbits, i.e. the points reached permuting the order of the two operations are different. This issue plays an important role in constructing algorithms for motion planning. In order to have commuting actions we may choose H to be the derived subgroup of Ω, i.e. the group generated by $\{\omega_1\omega_2\bar{\omega}_1\bar{\omega}_2 : \omega_1, \omega_2 \in \Omega\}$ and denoted $[\Omega, \Omega]$. Indeed for $[\Omega, \Omega]$ the following properties hold:

1. The derived subgroup of Ω, $[\Omega, \Omega]$, is a characteristic subgroup of Ω (i.e. it is stable for all automorphism of Ω). In particular it is a normal subgroup of Ω.

2. The quotient group $\Omega/[\Omega, \Omega]$ is an abelian group.

3. If $H \subset \Omega$ is a subgroup then the following are equivalent:

i) $H \supset [\Omega, \Omega]$;

ii) H is normal subgroup of Ω and Ω/H is commutative;

4. If Ω is generated by a set $\mathcal{B}$ of generators then $[\Omega, \Omega]$ is the normal subgroup generated by the set of commutators of elements of $\mathcal{B}$.

All these properties have been extensively used to treat the case of quantized chain form systems in [2]. It helps, for the analysis of reachable sets and planning, having a set of generators. From property **4.** above, we get the following:

Corollary 1 *Let Ω be a group generated by a set $\mathcal{B}$. Then $[\Omega, \Omega]$ is generated by the set of commutators of elements of $\mathcal{B}$ and if $\pi : \Omega \to \Omega/[\Omega, \Omega]$ is the canonical projection, then $\pi(\mathcal{B})$ generates $\Omega/[\Omega, \Omega]$.*

Let us now describe the link between normal subgroups and base–fiber decompositions. First we introduce some definitions and known results.

Definition 1 *Let H and G be two groups and $\tau : G \to Aut(H)$, $\tau(g) = \tau_g$, a homomorphism of G into the group of automorphisms of H, i.e. $\tau_g : H \to H$. The set $H \times G$ with the composition:*

$$\begin{array}{ccc} (H \times G) \times (H \times G) & \to & (H \times G) \\ ((h, g), (h', g')) & \mapsto & (h\tau_g(h'), gg') \end{array}$$

is called the external semi-direct product of G by H relative to τ and is denoted by $H \times_\tau G$.

Proposition 3 *The external semi-direct product $H \times_\tau G$ is a group. The mappings $i : H \to H \times_\tau G$, $i(h) = (h, e)$, $p : H \times_\tau G \to G$, $p(h, g) = g$, and $s : G \to H \times_\tau G$, $s(g) = (e, g)$ are group homomorphisms and s is a section, i.e. $p \circ s$ is the identity map from G to G.*

The following proposition gives a condition for the existence of a semi–direct decomposition.

Proposition 4 *Let Ω be a group, $H \subset \Omega$ a normal subgroup and $G \subset \Omega$ a subgroup such that $H \cap G = \{e\}$ and $HG = \{hg : h \in H, g \in G\} = \Omega$. Let $\tau : G \to Aut(H)$, $\tau(g) = \tau_g$ with $\tau_g(h) = gh\bar{g} \in H$. Then the map $(h, g) \mapsto hg$ is an isomomorphism of $H \times_\tau G$ onto Ω.*

In next proposition we show that a decomposition of Ω as a semidirect product induces a decomposition of the same type of the orbit $\mathcal{R}_q$ in base and fiber.

Proposition 5 *If $\Omega = H \times_\tau G$ satisfies the assumptions of Proposition 4 then we can decompose $\mathcal{R}_q$ into fiber $\mathcal{R}_q^H$ and base $\mathcal{R}_q^G$ in the following sense. The map $A_q : \Omega = H \times_\tau G \to \mathcal{R}_q$, defined by $A_q(\omega) = \mathcal{A}_\omega(q)$, is a bijection, satisfies $A_q((h, g)) = \mathcal{A}_{(\bar{g}hg, e)}\mathcal{A}_{(e,g)}(q)$, $A_q((H, e)) = \mathcal{R}_q^H$ and $A_q((e, G)) = \mathcal{R}_q^G$.*

Proof: Since the action of Ω is free, A_q is a bijection. Now, let $\omega = (h, g) \in H \times_\tau G$ then

$$(e, g)(\tau_{\bar{g}}h, e) = (e\tau_g\tau_{\bar{g}}h, g) = (h, g),$$

hence

$$\mathcal{A}_q(\omega) = \mathcal{A}_\omega(q) = \mathcal{A}_{(h,g)}(q) = \mathcal{A}_{(h',e)}\mathcal{A}_{(e,g)}(q),$$

where $h' = \tau_{\bar{g}}h = \bar{g}hg \in H$, since H is normal in Ω. ∎

An example of semi-direct decomposition is that of the group of isometries of the euclidean space $\mathbb{R}^n$, which decomposes as a semi-direct product of the group of parallel translations $GA(n)$ and the orthogonal group $O(n)$. In this case $\tau_R(t) = Rt \in GA(n)$ for all $R \in O(n)$ and $t \in GA(n)$, and the product in $GA(n) \times_\tau O(n)$ is given by the law: $(t_1, R_1)(t_2, R_2) = (t_1 + R_1 t_2, R_1 R_2)$. Recall that our concern is in quantized systems, hence we have to deal with subgroups, e.g. $\Omega \subset GA(n) \times_\tau O(n)$, and it is of interest to understand if it is possible to express a subgroup of a semidirect product as a semidirect product itself.

Proposition 6 *Let Ω be the external semi-direct product of G by H relative to τ, $H' \subset H$, $G' \subset G$ and $\tau_g(H') \subset H'$ for all $g \in G'$ Then the external semi-direct product of G' by H' relative to τ, $H' \times_\tau G'$ is a subgroup of Ω.*

Proof: We denote $\Omega' = H' \times_\tau G'$. Clearly any element of $(h, g) \in \Omega'$ also belong to Ω. Next we see that Ω' is a subgroup of Ω. The neutral element $(e, e) \in \Omega'$. Moreover, for each $(h, g) \in \Omega'$, $(\tau_{\bar{g}}\bar{h}, \bar{g})$ belongs to Ω' $(\tau_{\bar{g}}\bar{h} \in H')$ and is such that $(\tau_{\bar{g}}\bar{h}, \bar{g})(h, g) = (\tau_{\bar{g}}\bar{h}\tau_{\bar{g}}h, \bar{g}g) = (e, e)$. Finally $(h_1, g_1)(h_2, g_2) = (h_1\tau_{g_1}h_2, g_1, g_2) \in \Omega'$, since $\tau_{g_1}h_2 \in H'$. ∎

The converse is false. Think, for example, to the subgroup $\{(x, x) : x \in \mathbb{R}\}$ of the direct product $(\mathbb{R}, +) \times (\mathbb{R}, +)$. However we have the following

Proposition 7 *Let Ω be the external semi-direct product of G by H relative to τ, and $\Omega' \subset \Omega$ a subgroup. Let $H'' = i(H) \cap \Omega'$, $H' = i^{-1}(H'')$, $G'' = s(G) \cap \Omega'$, $G' = p(G'')$, then H' is a subgroup of H, G' is a subgroup of G, $\tau_g(H') \subset H'$ for all $g \in G'$, and $\Omega' \supset H' \times_\tau G'$.*

Proof: If $h_1, h_2 \in H'$, then $(h_1, e)(h_2, e) = (h_1h_2, e) \in \Omega'$, hence $h_1h_2 \in H'$. Take $h \in H'$ and let $(h_1, g_1) \in \Omega'$ be such that $(e, e) = (h, e)(h_1, g_1) = (h\tau_e h_1, g_1) = (hh_1, g_1)$, hence $g_1 = e$ and $h_1 \in H'$ is an inverse of h. Thus H' is a subgroup of H, similarly G' is a subgroup of G and $H' \times_\tau G' \subset \Omega'$. Now, for any $h_1, h_2 \in H'$, $g_1, g_2 \in G'$, $(h_1, g_1)(h_2, g_2) \in \Omega'$ since $(h_1, g_1), (h_2, g_2) \in \Omega'$. On the other hand $(h_1, g_1)(h_2, \bar{g}_1) = (h_1\tau_{g_1}h_2, e) \in \Omega' \cap i(H)$. Hence $\tau_{g_1}h_2 \in H'$. ∎

Having a set of generators of the acting group is a fundamental tool in the analysis of reachable sets for quantized systems. In particular, if the group is expressed as a semidirect product of G by H relative to τ, it is of help to individuate the generators of G and H. Therefore, assuming that Ω is freely generated by a subset $\mathcal{B}$ of Ω, we want to find sets of generators $\mathcal{B}_H$ and $\mathcal{B}_G$ for H and G respectively.

Proposition 8 *Let $\Omega = H \times_\tau G$, $\mathcal{B} \subset \Omega$ a set of generators, define*

$$\mathcal{B}_G = \{g : \exists h \in H \text{ s.t.} (h, g) \in \mathcal{B}\}$$

and

$$\mathcal{B}_H = \{\tau_{g_n \cdots g_1}h : g_i \in \mathcal{B}_G, \exists g \in G \text{ s.t.} (h, g) \in \mathcal{B}\}.$$

Then $\mathcal{B}_H$ and $\mathcal{B}_G$ generate H and G respectively.

Proof: Clearly any element of G is generated by a product of elements of $\mathcal{B}_G$. Moreover the group generated by $\mathcal{B}_H$ is contained in H. It remains to show that H is generated by $\mathcal{B}_H$. Let h be an element of H, then we can write

$$(h, e) = (h_n, g_n) \dots (h_1, g_1)$$

for some $n \in \mathbb{N}$ and $(h_i, g_i) \in \mathcal{B}$ for $i = 1, \dots, n$. We have $(h, e) = (h', g')$ with

$$h' = h_n(\tau_{g_n}h_{n-1})(\tau_{g_n}\tau_{g_{n-1}}h_{n-2}) \dots (\tau_{g_n} \dots \tau_{g_2}h_1)$$
$$g' = g_n g_{n-1} \dots g_1,$$

hence $g' = g_n g_{n-1} \dots g_1 = e$ and

$$h = h_n \prod_{i=n-1}^{1} \tau_{g_n} \dots \tau_{g_{i+1}} h_i.$$

Now, $\tau_{g_n} \dots \tau_{g_{i+1}} h_i \in \mathcal{B}_H$, i.e. $h \in H$ can be written as product of elements of $\mathcal{B}_H$. ∎

If Ω is finitely generated (i.e. $\mathcal{B}$ finite) then also $\mathcal{B}_G$ and $\mathcal{B}_H$ are finite and G and H are finitely generated.

Example 1. (continued) In the example of the polyedron rolling on a plane, each element $\omega \in \Omega$ can be identified with a pair $(t, \psi) \in \mathbb{R}^2 \times S^1$ and $\omega\omega' = (t, \psi)(t', \psi') = (t + R(\psi)t', \psi + \psi')$. Hence there is an isomorphism of $H \times_\tau G$ onto Ω, where H is the normal subgroup of elements that act only on the $\mathbb{R}^2$ component, with a translation that depends on the S^1 component, and G is the subgroup of elements that act only on the S^1 component. In [1] a finite set of generators of Ω is found, consequently a finite set of generators for H and G are obtained as in Proposition 8, a decomposition in base–fiber is operated, permitting a complete characterization of the structure of reachable sets.

4 Isometries and the n–trailer

If a distance d is defined on $\mathcal{Q}$ we may assume

(H6) $\forall \omega \in \Omega$, $\mathcal{A}_\omega$ **is an isometry.**

This means that $\forall q_1, q_2 \in \mathcal{Q}$, $d(\mathcal{A}_\omega(q_1), \mathcal{A}_\omega(q_2)) = d(q_1, q_2)$. Therefore the assumption **(H6)** implies (see Theorem 1 of [2]) that $\mathcal{R}_q$ is comprised either only of accumulation points or only of isolated points.
In general this assumption is quite restrictive, however, in case of base–fiber decomposition, we may have isometries in H (G) at least on the fiber $\mathcal{R}_q^H$ (base $\mathcal{R}_q^G$), as shown in next examples.

Example 1. (continued) Introduce on $\mathcal{Q} = \mathbb{R}^2 \times S^1$ the metric product of the Euclidean metric on $\mathbb{R}^2$ and the Riemannian metric on S^1 (inherited from $\mathbb{R}^2$), that is $d((x_1, \theta_1), (x_2, \theta_2)) = |x_1 - x_2| + \|\theta_1 - \theta_2\|$ where $\|\theta_1 - \theta_2\| = \min\{|\theta_1 - \theta_2 \pmod{2\pi}|, |\theta_2 - \theta_1 \pmod{2\pi}|\}$. With this metric $G \subset \mathrm{Iso}(\mathcal{Q}, d)$, the group of isometries, while the elements of H are not isometries. Indeed $d(\mathcal{A}_{(t,e)}(x_1, \theta_1), \mathcal{A}_{(t,e)}(x_2, \theta_2)) = |x_1 - x_2 + (R(\theta_1) - R(\theta_2))t| + \|\theta_1 - \theta_2\|$. However, they are isometries on the set of points having the same S^1 component, i.e. $H \subset \mathrm{Iso}(\mathcal{R}_q^H, d)$, for every $q \in \mathcal{Q}$.

Example 2. Sampled systems in chain form are treated in [2]. For those systems $\mathcal{Q} = \mathbb{R}^n$, H is the normal subgroup of actions that do not move the first two components and G is the subgroup that moves only the first two components. Here $\Omega = H \times G$ (direct product) are not isometries of $\mathbb{R}^n$ for the Euclidean metric d, but both H and G are subsets of $\mathrm{Iso}(\mathcal{Q}, d)$.

It may happen that G are isometries only on the base, that is $G \subset \mathrm{Iso}(\mathcal{Q}, d_G)$, where d_G is the pseudometric that measures the distance only on the base $\mathcal{R}_q^G$. This pseudometric is of particular interest in the case where the group G is not explicitely identified, but H is. Then we can consider the action of Ω/H on the set of H orbits, which may consist of isometries for d_G. Even more, this structure can be used also when **(H2)** does not hold, as shown in next example.

Example 3. Consider the n–trailer system, whose configuration space is given by $(x, y, \theta, \varphi_1, \ldots, \varphi_n)$, where $z = (x, y, \theta) \in \mathcal{Z} = \mathbb{R}^2 \times S^1$ gives the position and orientation of the leading car and $\varphi_i \in S^1$ is the relative angle formed by the $(i-1)$–th and the i–th trailers (the car is considered as 0 trailer). In case of $n = 1$, with constant speed of the car, the equations are given by

$$\begin{aligned} \dot{x} &= \cos(\theta) \\ \dot{y} &= \sin(\theta) \\ \dot{\theta} &= u_2 \\ \dot{\varphi} &= -\sin(\varphi) - u_2(1 + \cos(\varphi)). \end{aligned}$$

We focus on its quantized version obtained by sampling and taking a finite number of control inputs. In this case, assumption **(H2)** does not hold and the set Ω is only a monoid, not a group. However it is possible to identify the submonoid H of actions that do not move $\mathcal{Z}$. Let $\tilde{d}$ be the pseudometric which measure the distance on $\mathcal{Z}$ only, then the quotient $\mathcal{Q}/\tilde{d}$, obtained identifying elements with zero $\tilde{d}$ distance, is isometric to $\mathcal{Z}$. Identifying the elements of H with the empty word we obtain a set $\tilde{\Omega}$ that act as a group on $\mathcal{Z}$. This action is the same of the Dubins' car system and can be proved to be equivalent to that of rolling polyhedra extensively illustrated in Example 1. The natural decomposition of $\tilde{\Omega}$ has been explained above. It remains to study the action of H. For the variable $\psi = \tan(\frac{\varphi}{2})$, this action can be written in the form $\psi^+ = \lambda\psi + (1 - \lambda)$, where λ depends on the element of H. Linear systems of this kind were studied in [5] and we can use the same tools to understand the structure of the reachable set. The latter happens to be dense unless some resonance conditions are verified.

References

[1] A. Bicchi, Y. Chitour, and A. Marigo. Reachability and steering of rolling polyhedra: A case study in discrete nonholonomy. Submitted to IEEE Trans. on Aut. Control, 2002.

[2] A. Bicchi, A. Marigo, and B. Piccoli. On the recahability of quantized control systems. *IEEE Trans. on Automatic Control*, 47(4):546–563, 2002.

[3] R. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Autom. Control*, 45(7):1279–1289, July 2000.

[4] M. Ceccarelli, A. Marigo, S. Piccinocchi, and A. Bicchi. Planning motions of polyhedral parts by rolling. *Algorithmica*, 26(4):560–576, 2000.

[5] Y. Chitour and B. Piccoli. Controllability for discrete systems with a finite control set. *Math. Control Signals Systems*, 14(2):173–193, 2001.

[6] D. F. Delchamps. Extracting state information from a quantized output record. *Systems and Control Letters*, 13:365–371, 1989.

[7] D. F. Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE Trans. Autom. Control*, 35(8):916–926, 1990.

[8] N. Elia and S. Mitter. Stabilization of linear systems with limited information. *IEEE Trans. on Automatic Control*, 46(9):1384–1400, September 2001.

[9] A. Marigo, B. Piccoli, and A. Bicchi. Reachability analysis for a class of quantized control systems. In *Proc. IEEE Int. Conf. on Decision and Control*, pages 3963–3968, 2000.