GENERALIZED FIBONACCI DYNAMICAL SYSTEMS

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Abstract. In this paper we consider generalizations of dynamical systems that are based on the Fibonacci sequence. We first study stability properties of such systems for both the continuous and discrete–time case. Then, by considering the Kronecker operator, a further class of dynamical systems is introduced whose outputs can be used to define possible generalization of the golden section. Applications of such system may range from realization of digital filters, manufacturing of tissue with fractal property, etc. Properties of sequences generated by these systems are partially considered and has to be further addressed.

1. Introduction

There is a large amount of literature on Fibonacci numbers and their generalizations. Biographical information about Fibonacci can be found at the MacTutor History of Mathematics Archive at [1]. A compendium of information about the Fibonacci numbers can be found at [2], whereas their applications to art, architecture, and music can be found at [2]. A quarterly journal since 1963 is dedicated to research related to Fibonacci numbers and related questions: the website for the Fibonacci Quarterly can be found at [3]. Also useful are the books in [4–8]. A large variety of generalizations is available including Fibonacci polynomials, tribonacci numbers, k–nacci or multinacci numbers [9, 10]. Many properties of Fibonacci and related sequences are discussed in [11].

In this paper we consider generalizations of dynamical systems that are based on the Fibonacci sequence. We first study stability properties of such systems for both the continuous and discrete–time case. Then, by considering the Kronecker operator, we introduce a further class of dynamical systems whose outputs can be used to define possible generalization of the golden section. Applications of such system may range from realization of digital filters, manufacturing of tissue with fractal property, etc. Properties of sequences generated by these systems are partially considered and has to be further addressed.

2. The discrete–time case

The Fibonacci sequence, $\{f_i\}$, for $i = 0, 1, \ldots$, is described by the iterative rule:

$$\begin{cases} f_{k+2} = f_{k+1} + f_k \,, \\ f_0 = 0 \,, \\ f_1 = 1 \,, \end{cases} \tag{2.1}$$

but may be considered also as the impulsive response of the basic discrete–time *Fibonacci dynamic system*:

$$\begin{aligned} x(k+1) &= F\, x(k) + b\, u(k) \,, \\ y(k) &= C\, x(k) \,, \end{aligned} \tag{2.2}$$

where $x \in \mathbb{R}^2$ is the system state, $u \in \mathbb{R}$ is the system input, $y \in \mathbb{R}$ is the system output, and

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{2.3}$$

Indeed, starting from the state origin, i.e. $x(0) = (0,0)^T$, and applying the impulsive input:

$$u(k) = \begin{cases} 1 & k = 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.4}$$

the state $x(k)$ evolves as follows:

$$\begin{array}{c} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \end{pmatrix}, \\ \begin{pmatrix} 8 \\ 13 \end{pmatrix}, \begin{pmatrix} 13 \\ 21 \end{pmatrix}, \begin{pmatrix} 21 \\ 34 \end{pmatrix}, \begin{pmatrix} 34 \\ 55 \end{pmatrix}, \begin{pmatrix} 55 \\ 89 \end{pmatrix}, \ldots \end{array} \tag{2.5}$$

and the system output $y(k)$ is indeed the Fibonacci sequence. Clearly, starting with different initial states $x(0)$, other sequences are obtained. E.g. with $x(0) = (2,1)^T$, we obtain the Lucas' sequence [2].

In systems theory, the transfer function $G(z)$ of a given time–invariant linear system is introduced to characterize the relation between the system input $u(k)$ and the corresponding output $y(k)$. For a discrete–time system, the transfer function $G(z)$ can be computed as the ratio between the Z–transform of the input signal $u(k)$ and the Z–transform of the output signal $y(k)$. For a discrete–time signal $x(k)$ being defined only for $k \geq 0$, its (unilateral) Z–transform $X(z)$ is defined as follows [12]:

$$X(z) = \sum_{k=0}^{\infty} x(k)\, z^{-k}\,. \tag{2.6}$$

The transfer function $G(z)$ of the system in Eq. 2.2 can easily be obtained by direct computation. This gives the following result:

$$G(z) = C\,(z\,I - F)^{-1}\,b = \frac{1}{z^2 - z - 1}\,. \tag{2.7}$$

Moreover, given the basic Fibonacci matrix F, its k–th power is:

$$F^k = \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix}, \tag{2.8}$$

that is very useful in computing the explicit evolution of the system (2.2).

Furthermore, the same approach applies to so–called *tetranacci* and even to *k–nacci* systems. A very simple generalization of the system (2.2) is indeed the following:

$$\begin{aligned} x(k+1) &= \alpha(k) \begin{pmatrix} 0 & I_{n-1} \\ 1 & 1_{n-1} \end{pmatrix} x(k) + \begin{pmatrix} 0_{n-1} \\ 1 \end{pmatrix} u(k)\,, \\ y(k) &= C\,x(k)\,, \end{aligned} \tag{2.9}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}$ is the system input, and $y(k)$ is the system output. Moreover, given the matrix

$$F_n = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 1_{n-1} \end{pmatrix}, \tag{2.10}$$

the choice

$$|\alpha(k)\lambda_{\max}(F_n)| < 1, \tag{2.11}$$

assures the asymptotic stability of the system (2.9). Furthermore, vector C may be chosen as

$$C = \begin{pmatrix} \gamma & 0_{n-1} \end{pmatrix} \tag{2.12}$$

to realize the following transfer function:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\gamma\,\alpha^{n-1}}{z^n - \alpha z^{n-1} \cdots - \alpha^{n-1}\,z - \alpha^n}\,, \tag{2.13}$$

or as

$$C = \begin{pmatrix} \delta & \delta & 0_{n-2} \end{pmatrix} \tag{2.14}$$

to realize the following transfer function with enhanced low–pass characteristics:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\delta\,\alpha^{n-1}(z+1)}{z^n - \alpha\,z^{n-1} \cdots - \alpha^{n-1}\,z - \alpha^n}\,. \tag{2.15}$$

The parameter δ is chosen so that the filter gain is 1, i.e. $G(1) = 1$.

Moreover, matrix F_n is positive, and thus the existence of a dominant eigenvalue — and of the corresponding positive eigenvector — is assured by Perron–Frobenius Theorem [13]. An estimation of the maximum eigenvalue is given by:

$$\lambda_{\max}(F_n) \simeq 2 - \frac{\Phi - 1}{\Phi^n - 1} = 2 - \frac{\varphi}{\Phi^n - 1}\,, \tag{2.16}$$

where Φ is the golden section and is given by:

$$\Phi = 1 + \varphi = \frac{1+\sqrt{5}}{2}\,. \tag{2.17}$$

3. The continuous–time case

The same procedure allows us to introduce generalized Fibonacci systems in the continuous–time case. Indeed, the eigenstructure of Fibonacci systems is retained by the following class of models:

$$\begin{aligned} \dot{x}(t) &= \left(-\beta\,I + \alpha \begin{pmatrix} 0 & I_{n-1} \\ 1 & 1_{n-1} \end{pmatrix}\right) x(t) + \begin{pmatrix} 0_{n-1} \\ 1 \end{pmatrix} u(t)\,, \\ y(t) &= \begin{pmatrix} \gamma & 0_{n-1} \end{pmatrix} x(t)\,, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}$ is the system input, $y \in \mathbb{R}$ is the system output, and $\alpha, \beta \in \mathbb{R}$ are parameters to be chosen. Clearly, we are interested in those models of this class that are asymptotically stable, which is guaranteed by the following choice:

$$\beta > |\alpha\,\lambda_{max}(F_n)|\,. \tag{3.2}$$

For the case with $n = 2$, given an initial state $x(0)$, the evolution of the unforced system, i.e. the state trajectory obtained when $u(t) = 0$, is:

$$x(t) = \frac{e^{-\beta t}}{1+2\varphi} \begin{pmatrix} 1 & 1 \\ 1+\varphi & -\varphi \end{pmatrix} \begin{pmatrix} e^{\alpha(1+\varphi)\,t} & 0 \\ 0 & e^{-\varphi\alpha t} \end{pmatrix} \begin{pmatrix} \varphi & 1 \\ 1+\varphi & -1 \end{pmatrix} x(0)\,. \tag{3.3}$$

The transfer function $G(s)$ of a continuous–time system characterizes the relation between the system input $u(t)$ and its output $y(t)$. This can be computed as the ratio between the Laplace transform of $u(t)$ and that of $y(t)$. For a continuous–time signal $x(t)$ being defined for $t \geq 0$ its Laplace transform $X(s)$ is defined as [12]:

$$X(s) = \int_0^\infty x(t)\, e^{-st} dt\,. \tag{3.4}$$

In our case, the transfer function $G(s)$ is given by:

$$\begin{aligned} G(s) = \frac{Y(s)}{U(s)} &= \frac{\gamma\, \alpha^2}{(s+\beta)^2 - \alpha\,(s+\beta) - \alpha^2} = \\ &= \frac{\beta^2 - \alpha\,\beta - \alpha^2}{s^2 + (2\beta - \alpha)\, s + \beta^2 - \alpha^2 - \alpha\,\beta}\,, \end{aligned} \tag{3.5}$$

where γ is chosen so that $G(0) = 1$. The system poles can be computed as the roots of the transfer function's denominator and are given by:

$$\begin{aligned} \lambda_1 &= -\beta + \alpha\, \Phi\,, \\ \lambda_2 &= -\beta - \alpha\, \varphi\,. \end{aligned} \tag{3.6}$$

Investigation of properties of digital as well as analogous filters based on Fibonacci structure are out of the scope of this work and can be found in [14].

4. Generalizations via Kronecker products and sums

Kronecker products and sums can be used to introduce further generalizations of Fibonacci systems. Consider a discrete–time system with a dynamic matrix A that is described by the following Kronecker product:

$$A = \alpha\, \left(F_n \otimes F_m\right)\,, \tag{4.1}$$

where $F_n \in \mathbb{R}^{n\times n}$ and $F_m \in \mathbb{R}^{m\times m}$, and $\alpha \in \mathbb{R}$ is a given constant that have to be fixed.

Since we know that the maximum eigenvalue of F_k is estimated as in Eq. 2.16, then we may estimate the maximum eigenvalue of the system matrix A. Indeed we have:

$$\lambda_{\max}(A) \simeq \alpha \left(2 - \frac{\Phi - 1}{\Phi^n - 1}\right) \left(2 - \frac{\Phi - 1}{\Phi^m - 1}\right)\,. \tag{4.2}$$

Therefore the discrete–time system

$$\begin{aligned} x(k+1) &= \alpha\, \left(F_n \otimes F_m\right)\, x(k) + \begin{pmatrix} 0_{n+m-1} \\ 1 \end{pmatrix} u(k)\,, \\ y(k) &= \begin{pmatrix} \gamma & 0_{n+m-1} \end{pmatrix} x(k)\,, \end{aligned} \tag{4.3}$$

where $x \in \mathbb{R}^{n+m}$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, by choosing

$$|\alpha \lambda_{\max}(A)| < 1, \tag{4.4}$$

is asymptotically stable. Furthemore, a choice such that $\alpha > 0$ makes the realization also positive.

The analysis of the dynamics is greatly simplified by taking into account the following property [13]:

$$\left(\alpha\, \left(F_n \otimes F_m\right)\right)^k = \alpha^k \left({F_n}^k \otimes {F_m}^k\right)\,. \tag{4.5}$$

By using the same procedure, generalized Fibonacci systems in the continuous–time case can also be introduced. Indeed, the eigenstructure of Fibonacci systems is generalized by the following class of models:

$$\begin{array}{rcl} \dot{x}(t) & = & (-\beta\, I + \alpha\, (F_n \otimes F_m))\, x(t) + \begin{pmatrix} 0_{n+m-1} \\ 1 \end{pmatrix} u(t)\,, \\ y(t) & = & \begin{pmatrix} \gamma & 0_{n+m-1} \end{pmatrix} x(t)\,. \end{array} \tag{4.6}$$

The choice of

$$\beta > \alpha \left(2 - \frac{\Phi - 1}{\Phi^n - 1}\right)\left(2 - \frac{\Phi - 1}{\Phi^m - 1}\right), \tag{4.7}$$

assures the asymptotic stability of the system.

4.1. **The Multi–dimensional Case.** Consider the following class of discrete–time autonomous systems with multi–dimensional state X_k that evolves according to the following rule:

$$\begin{array}{rcl} X_{k+1} & = & (\gamma\, F) \otimes X_k\,, \\ y_k & = & g(X_k)\,, \end{array} \tag{4.8}$$

where y_k is the system output, $\gamma \in \mathbb{R}$ can be interpreted as a *fading factor*, F is the Fibonacci matrix, $\otimes$ is the Kronecker matrix operator, and g is an output function.

Given an initial state value $X(0)$ ($X(0) \in \mathbb{R}^{p(0)\times q(0)}$), the system evolution evolves according to:

$$X_k = (\gamma\, F \otimes)^k\; X(0)\,, \tag{4.9}$$

where $X_k \in \mathbb{R}^{p(k)\times q(k)}$, with $p(k) = 2^k\, p(0)$ and $q(k) = 2^k\, q(0)$, and the k–th power is to be interpreted in the Kronecker sense. The first values of the state sequence are the following:

$$\begin{array}{rcl} X_0 & = & X(0)\,, \\ X_1 & = & \begin{pmatrix} 0 & X(0) \\ X(0) & X(0) \end{pmatrix}, \\ X_2 & = & \begin{pmatrix} 0 & X_1 \\ X_1 & X_1 \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & X(0) \\ 0 & 0 & X(0) & X(0) \\ \hline 0 & X(0) & 0 & X(0) \\ X(0) & X(0) & X(0) & X(0) \end{array}\right). \end{array}$$

The initial state value $X(0)$ can be interpreted as a *base pattern*, or *seed*, that is somehow repeated so as to *grow* a more complex object. Fig. 1 shows a graphical representation of the system state after 5 steps by starting from $X(0) = 1$ and $X(0) = [1, 0; 1, 1]$, respectively. Elements of X_5 that are equal to 1 are represented as circles, whereas null elements are represented as blank space. A *fractal* structure is clearly revealed in the figure.

Furthermore, one possible choice for the output function g is the row sum of X_k, i.e.

$$y_k = \begin{pmatrix} \sum_j X_k(1, j) \\ \vdots \\ \sum_j X_k(p(k), j) \end{pmatrix}. \tag{4.10}$$

Clearly, we have $y_k \in \mathbb{R}^{p(k)}$. Let us define vector σ as the column sum of $X(0)$, i.e. $\sigma^i = \sum_j X(0)(i, j)$. Clearly, we have $\sigma \in \mathbb{R}^{p(0)}$. The first values of the output

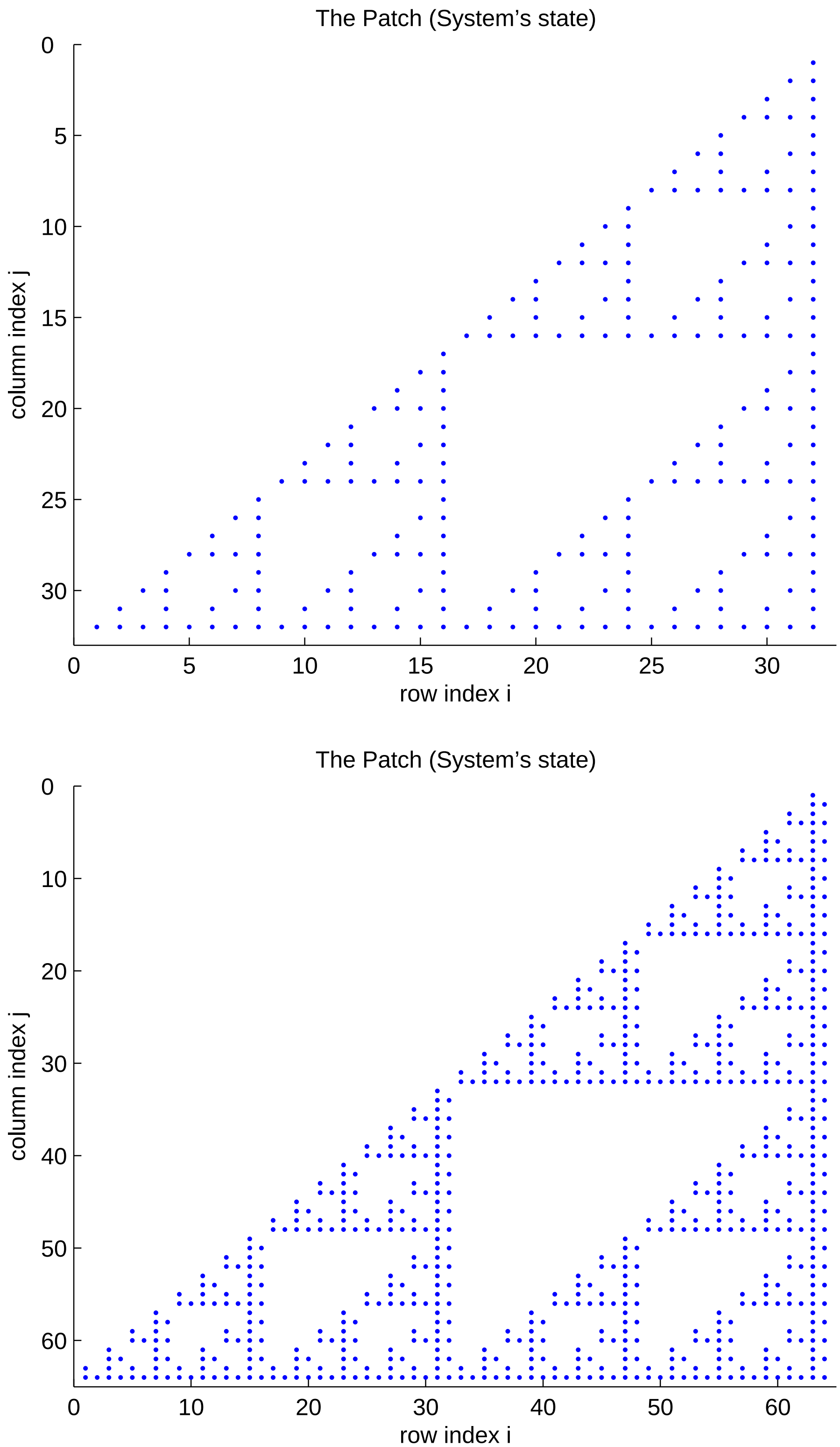

FIG. 1. Graphical representation of the system state after 5 steps by starting from $X(0) = 1$ and $X(0) = [1, 0; 1, 1]$, respectively. Elements of X_5 that are equal to 1 are represented as circles, whereas null elements are represented as blank space.

sequence are the following:

$$\begin{aligned} y_0 &= \sigma\,, \\ y_1 &= \begin{pmatrix} y_0 \\ 2\,y_0 \end{pmatrix} = \begin{pmatrix} \sigma \\ 2\,\sigma \end{pmatrix}\,, \\ y_2 &= \begin{pmatrix} y_1 \\ 2\,y_1 \end{pmatrix} = \begin{pmatrix} \sigma \\ 2\,\sigma \\ 2\,\sigma \\ 4\,\sigma \end{pmatrix}\,. \end{aligned}$$

Indeed, it can be proved that the following the recurrent formula holds:

$$y_k = \begin{pmatrix} y_{k-1} \\ 2\,y_{k-1} \end{pmatrix}\,. \tag{4.11}$$

4.1.1. *The Integer Case.* An interesting interpretation of the behavior of systems of Eq. 4.8 can be obtained by considering evolutions starting from non–negative integer initial states $X(0) \in \mathbb{N}^{p(0)\times q(0)}$, i.e. where the elements of the matrix $X(0)$ are non-negative integers. Indeed, the systems can be considered as generators of integer output sequences. As examples, let us assume $\gamma = 1$ and consider the system output after $k = 3$ steps for varying initial states $X(0)$. According to (4.10), the output is given by the following vectors:

$$\begin{aligned} y_3(X(0)=0) &= (0,\dots,0)^T\,, \\ y_3(X(0)=1) &= (1,2,2,4,2,4,4,8,2,4,4,8,4,8,8,16)^T\,, \\ y_3(X(0)=[0,1]^T) &= (1,0,2,0,2,0,4,0,2,0,4,0,4,0,8,0,4,0,8,0,8,0,16,0)^T\,, \\ y_3(X(0)=[1,0]^T) &= (0,1,0,2,0,2,0,4,0,2,0,4,0,4,0,8,0,4,0,8,0,8,0,16)^T\,, \\ y_3(X(0)=[1,1]^T) &= (1,1,2,2,2,2,4,4,2,2,4,4,4,4,8,8,4,4,8,8,8,8,16,16)^T\,. \end{aligned} \tag{4.12}$$

Under the hypothesis that $X(0) \neq 0$, consider vector $z_k = (\alpha_1, \dots, \alpha_{p(k)})^T$, where α_i is obtained by summing the elements of y_k group–wise from index $(i-1)\,p(0)$ to index $i\,p(0)$. Elements of z_k represent another positive integer sequence. With reference to the initial states of the examples in Eq. 4.12, we obtain:

$$\begin{aligned} z_3(X(0)=1) &= (1,2,2,4,2,4,4,8,2,4,4,8,4,8,8,16)^T\,, \\ z_3(X(0)=[0,1]^T) &= (1,2,2,4,2,4,4,8,2,4,4,8,4,8,8,16)^T\,, \\ z_3(X(0)=[1,0]^T) &= (1,2,2,4,2,4,4,8,2,4,4,8,4,8,8,16)^T\,, \\ z_3(X(0)=[1,1]^T) &= (2,4,4,8,4,8,8,16,4,8,8,16,8,16,16,32)^T\,, \end{aligned} \tag{4.13}$$

It is worth observing that such sequences present similar repeating patterns. Furthermore, at any discrete time k, consider the following approximation of the *continued fraction*:

$$s_k = \alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{1}{\alpha_4 + \cfrac{1}{\ddots \; \cfrac{1}{\alpha_{p(k)}}}}}} . \tag{4.14}$$

Given an initial state $X(0)$, the sequence $\{s_k\}$ converges to a number that depends on the initial state itself. In particular, we have the following examples:

$$\begin{aligned} s(X(0)=1) &= (1.00000, 1.5000, 1.40000, 1.40909, 1.408163, 1.40825, \dots)\ , \\ s(X(0)=2) &= (1.0000, 2.2500, 2.2353, 2.2357, 2.2357, 2.2357, \dots)\ . \end{aligned} \tag{4.15}$$

There is a large amount of studies in the literature on number theory that consider generalizations of the golden section via the continued fraction [15]. In this vein, we can interpret the sequences of Eq. 4.15 as other possible generalization of the golden section Φ. To this aim, the repeating patterns of the sequence z_k must be further investigated.

5. Conclusion

Generalization of dynamical systems based on the Fibonacci sequence was presented in this work. Stability of such systems have been studied for both the continuous and discrete–time case. A further generalization of dynamical system based on the Kronecker operator and the Fibonacci sequence was also presented. This class contains systems generating converging sequences, that are related to the golden section. Properties of these sequences merit further investigations and will be studied in future work.

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