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CONTROL SYNTHESIS FOR PRACTICAL STABILIZATION OF QUANTIZED LINEAR SYSTEMS*

Abstract. In this work we face the stability problem for quantized control systems (QCS). A discrete–time single–input linear model is considered and, motivated by technological applications, we assume that a uniform quantization of the control set is a priori fixed. As it is well known, for QCS only practical stability properties can be achieved, therefore we focus on the existence and construction of quantized controllers capable of steering a system to within invariant neighborhoods of the equilibrium.
The main contribution of the paper consists in a theorem which provides a condition for the practical stabilization in a fixed number of steps: not only the result is interesting in itself, but also it enables to construct a family of stabilizing controllers by means of Model Predictive Control (MPC) techniques.
In the last part of the paper some results on the characterization of controlled–invariant sets are reviewed and a lower bound on the size of invariant sets is provided. The bound is attained by an explicitly constructed element.

1. Introduction

The interest of the control community for quantized control systems (QCS) has been considerably raising in the past twenty years. Situations in which quantization may arise and can not be neglected are varied: a popular example is that of "networked control systems", i.e., systems interconnected through communication channels capable of transmitting only a finite amount of information between the plant and the controller.

Special attention has been devoted to the stabilization problem for QCS (see for instance [5, 6, 7, 9, 10, 11, 13, 14, 19, 21]): in [6] the author clarifies that asymptotic stability is a too strong requirement for QCS, hence practical stability concepts have been considered.

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Unlike most of the existing literature (where quantization is considered as a parameter to be designed), our work is inspired by the belief that another kind of question is as much important: the stability problem for systems whose quantized resources (i.e., discrete input and output sets) are *fixed a priori*. Such analysis is helpful because it allows to decide in advance whether a desired control objective can be achieved by using a *given* technology (actuators, sensors, communication and computational means). Moreover, issues of this kind may also represent the basis to solve more general stabilization problems (remarkable examples are presented in [9]).

This paper is focused on the stabilization of single–input discrete–time linear systems where a uniformly quantized control set is given.
Let Ω and X_0 be two neighborhoods of the origin with $\Omega \subseteq X_0$: the aim is to design (X_0, Ω)–stabilizing controllers, that is feedback control laws rendering both Ω and X_0 invariant and such that all the states of X_0 are initial points of trajectories which enter Ω in a finite time.
Since the control set is given, the problem is not feasible for all pairs (X_0, Ω). Previous results on the construction of invariant neighborhoods obtained in [15, 14] are useful in this context and briefly reported in Section 3.1 . In particular, a continuous family of invariant sets that includes a minimal element called $Q_n(\epsilon)$ is constructed. In the same section we review necessary and sufficient conditions on the control set diameter ensuring that the system is $\big(X_0, Q_n(\epsilon)\big)$–stabilizable.

Section 3.2 is the core of this work and contains the main original contribution: the $\big(X_0, Q_n(\epsilon)\big)$–stabilizability problem enforcing a bound on the number of steps to converge within $Q_n(\epsilon)$ is addressed. A sufficient (and in some cases necessary) condition is provided on the diameter of the control set ensuring the desired stability property. This result is interesting in itself and it is also a useful tool for establishing feasibility of optimal control problems. This leads to the construction of a family of stabilizing controllers by application of Model Predictive Control (MPC) techniques.
Since the goal is the stabilization of the system near the origin, we are interested in confining trajectories within small controlled–invariant neighborhoods of 0 . Hence, in Section 4 , we review the minimality properties of $Q_n(\epsilon)$ (proved in [14]) and present an alternative proof of the main result on the subject.

Notation: $Q_n(\Lambda) := \big[-\frac{\Lambda}{2}; \frac{\Lambda}{2}\big]^n = \big\{x \in \mathbb{R}^n \,|\, \|x\|_\infty \le \frac{\Lambda}{2}\big\}$ is the hypercube of edge length Λ whilst $Q_n^o(\Lambda) := \big[-\frac{\Lambda}{2}; \frac{\Lambda}{2}\big)^n$ is the semi–open hypercube. $\lfloor x \rfloor :=$ max $\{z \in \mathbb{Z} \,|\, z \le x\}$ and $\lceil x \rceil :=$ min $\{z \in \mathbb{Z} \,|\, z \ge x\}$ are the floor and the ceil function. ${}^cE$ denotes the complementary of E , $-E := \{x \in \mathbb{R}^n \,|\, -x \in E\}$, $\mathrm{diam}(E) := \sup\big\{\|x-y\|_2 \,|\, (x,y) \in E \times E\big\}$ is the diameter of E , $\mathrm{Pr}_i\, x := x_i$ is the projection on the i^{th} coordinate axis and $\mathrm{diam}_i\, \Omega := \mathrm{diam}\big(\mathrm{Pr}_i\, \Omega\big)$. $|A|$ is the matrix defined by $|A|_{i,j} := |A_{i,j}|$, x' denotes the transpose of the vector x , $Q = Q' > 0$ means that Q is a symmetric positive definite matrix. $x^+(t)$ denotes $x(t+1)$: the dependance on t will be often omitted.

2. Preliminaries

We deal with a single–input discrete time–invariant linear system subject to a fixed uniformly quantized control set, more precisely:

$$(1) \qquad \begin{cases} x^+(t) = Ax(t) + bu(t) \\ x \in \mathbb{R}^n, \quad u \in \mathcal{U} \subseteq \epsilon\mathbb{Z} \quad (\epsilon > 0) \\ A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n, \end{cases}$$

We suppose that the pair (A, b) is reachable. In this case, changing the coordinates in the state space, we can assume
H1) the pair (A, b) is in *controller form*:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

where $s^n - \alpha_n s^{n-1} - \cdots - \alpha_2 s - \alpha_1$ is the characteristic polynomial of A.
If $\sum_{i=1}^n |\alpha_i| < 1$, the system is asymptotically stable, we hence assume
H2) $\sum_{i=1}^n |\alpha_i| \geq 1$.
Let us recall the basic definitions about invariant sets [4]:

Definition 1. *The set $\Omega \subseteq \mathbb{R}^n$ is said to be* positively invariant *for a closed–loop system $x^+ = f(x)$ iff $\forall\, x \in \Omega$, $x^+ \in \Omega$;*

Definition 2. *The set $\Omega \subseteq \mathbb{R}^n$ is said to be* controlled–invariant *for system (1) iff $\forall\, x \in \Omega\ \exists\, u \in \mathcal{U}$ such that $x^+ = Ax + bu \in \Omega$.*

The weak (practical) stability notion we will use is the (X_0, Ω)–*stability* (see also [9]):

Definition 3. *Let $0 \in \Omega \subseteq X_0 \subseteq \mathbb{R}^n$ with Ω being a neighborhood of 0; a feedback law $u : \mathbb{R}^n \to \mathcal{U}$ is said to be (X_0, Ω)–*stabilizing *iff it renders both Ω and X_0 positively invariant and $\forall\, x(0) \in X_0\ \exists\, t_{x(0)} \in \mathbb{N}$ such that $x(t_{x(0)}) \in \Omega$.*
*If moreover $\forall\, x(0) \in X_0\ t_{x(0)} \leq H_p$, then the feedback is said to be (X_0, Ω)–*stabilizing in H_p steps.
*System (1) is said to be (X_0, Ω)–*stabilizable (in H_p steps) *iff there exists an (X_0, Ω)–stabilizing (in H_p steps) feedback law.*

3. Invariant sets and stabilizing control laws

3.1. Review

We briefly review some basic results concerning the practical stabilization problem: for a more detailed treatment we refer to [14].

THEOREM 1. *If* $\mathcal{U} = \epsilon\mathbb{Z}$, *then* $\forall H_p \geq n$ *and* $\forall \Delta \geq \epsilon$, *system (1) is* $\big(Q_n(\Delta)\,,\, Q_n(\epsilon)\big)$*–stabilizable in* H_p *steps.* □

Theorem 1 holds for arbitrarily large Δ's because the control set is unbounded. In the finite control set case we have analyzed the invariance and stabilizability properties for control sets of the type $\mathcal{U}_k := \{-k\epsilon, \dots, 0, \dots, +k\epsilon\}$, $\forall k \in \mathbb{N}$. Let us introduce the *saturated* quantized deadbeat controllers:

DEFINITION 4. *Let* $k \in \mathbb{N}$ *and*

$$w(x) := \begin{cases} -k\,\epsilon & \text{if } \left\lfloor -\frac{\sum_{i=1}^n \alpha_i x_i}{\epsilon} + \frac{1}{2} \right\rfloor < -k \\ \left\lfloor -\frac{\sum_{i=1}^n \alpha_i x_i}{\epsilon} + \frac{1}{2} \right\rfloor \cdot \epsilon & \text{otherwise}. \end{cases}$$

The feedback law $u : \mathbb{R}^n \to \mathcal{U}_k$ *defined by*

$$\begin{cases} u(x) = w(x) & \text{if } \sum_{i=1}^n \alpha_i x_i \geq 0 \\ u(x) = -w(-x) & \text{otherwise}. \end{cases} \tag{2}$$

is called the k–levels saturated *quantized deadbeat controller* $\big([k]\,qdb$*–controller*$\big)$. *Denote by* Ξ *the region where the controller saturates, namely* $\Xi = \Xi_1 \cup (-\Xi_1)$, *where*

$$\Xi_1 := \left\{ x \in \mathbb{R}^n \,\middle|\, \left\lfloor -\frac{\sum_{i=1}^n \alpha_i x_i}{\epsilon} + \frac{1}{2} \right\rfloor < -k \right\} = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n \alpha_i x_i > k\epsilon + \frac{\epsilon}{2} \right\}.$$

LEMMA 1. *Let* $k \in \mathbb{N}$, *consider the closed–loop dynamics induced by the* $[k]\,qdb$*–controller, then*

$$|x_n^+| \leq \frac{\epsilon}{2} \iff x \notin \Xi \iff \left| \sum_{i=1}^n \alpha_i x_i \right| \leq \frac{\epsilon}{2} + k\epsilon\,. \qquad \square$$

Consider system (1), assume that $\Delta \geq \epsilon$ and let

$$k(\Delta) := \left\lceil \frac{1}{2}\frac{\Delta}{\epsilon}\Big(\sum_{i=1}^n |\alpha_i| - 1 \Big) \right\rceil. \tag{3}$$

PROPOSITION 1. *Assume that* $\mathcal{U} = \mathcal{U}_k$ *and* $\Delta \geq \epsilon$, *the following properties are equivalent:*
i) $Q_n(\Delta)$ *is controlled–invariant;*
ii) $k \geq k(\Delta)$;
iii) $Q_n(\Delta)$ *is positively invariant for the closed–loop system* $x^+ = Ax + b\,u(x)$, *where* $u(x)$ *is the* $[k]\,qdb$*–controller.* □

The basic result concerning the stabilizability analysis is

PROPOSITION 2. *Let* $\Delta > \epsilon$ *, consider* $k(\Delta)$ *as in Equation (3) and*

$$\overline{k} := \begin{cases} k(\Delta) & \textit{if} \quad \frac{1}{2}\frac{\Delta}{\epsilon}\big(\sum_{i=1}^{n} |\alpha_i| - 1 \big) \notin \mathbb{N} \\ k(\Delta)+1 & \textit{otherwise}. \end{cases}$$

The $[\,\overline{k}\,]$*qdb–controller is* $\big(Q_n(\Delta)\,,\,Q_n(\epsilon)\big)$*–stabilizing.* □

One could expect that, in order to achieve the $\big(Q_n(\Delta)\,,\,Q_n(\epsilon)\big)$–stability, it is necessary a control set diameter larger than the one ensuring the invariance of $Q_n(\Delta)$. On the contrary Proposition 2 shows that, generically, such diameter is also sufficient for the $\big(Q_n(\Delta)\,,\,Q_n(\epsilon)\big)$–stabilizability. This phenomenon is referable to the control quantization: in fact, the minimal diameter of the control set ensuring the invariance of $Q_n(\Delta)$ is larger than the one necessary in the continuous case (because of the ceil function) so that the controller has enough authority to achieve also the convergence towards $Q_n(\epsilon)$.

3.2. Synthesis of stabilizing control laws: Model Predictive Control

The rather strong property of the control set described in Proposition 2, to be both necessary and generically sufficient for $\big(Q_n(\Delta)\,,\,Q_n(\epsilon)\big)$–stabilizability, has as a counterpart a relative weakness, in that no bound on the number of steps necessary to reach $Q_n(\epsilon)$ can be enforced. On the other hand, it is natural to expect that a larger control set would ensure better performance in terms of convergence time. We are hence interested in looking for conditions on the control set diameter for the $\big(Q_n(\Delta)\,,\,Q_n(\epsilon)\big)$–stabilizability in a fixed number of steps.

THEOREM 2. *Let* $\Delta > \epsilon > 0$ *and* $\mathcal{U} \subseteq \epsilon\mathbb{Z}$*. Fix* $H_p \geq n:\;\; H_p = n+p-1$ *with* $p \geq 1$*. A sufficient condition in order that the system is* $\big(Q_n(\Delta)\,,\,Q_n(\epsilon)\big)$*–stabilizable in* H_p *steps is that* $\mathcal{U}_{k_p} \subseteq \mathcal{U}$*, with*

$$k_p = \left\lceil \frac{1}{2}\frac{\Delta}{\epsilon}\Big(\sum_{i=1}^{n} |\alpha_i| - 1 \Big) + \frac{1}{\epsilon}\frac{\Delta-\epsilon}{2\psi_p} \right\rceil, \tag{4}$$

where the sequence $\{\psi_m\}_{m\in\mathbb{N}\setminus\{0\}}$ *is defined as follows:*

$$\begin{cases} \psi_1 := 1 \\ \psi_m := 1 + \sum_{i=1}^{m-1} |\alpha_{n-m+i+1}|\,\psi_i\,, \quad m \geq 2\,, \\ \qquad\quad \textit{where} \;\; \alpha_j := 0 \;\; \textit{if} \;\; j \leq 0\,. \end{cases} \tag{5}$$

Moreover, if $\alpha_i \geq 0\; \forall\, i = 1,\ldots,n$*, the bound* k_p *is strict, that is* $\mathcal{U}_{k_p-1}$ *does not make the system* $\big(Q_n(\Delta)\,,\,Q_n(\epsilon)\big)$*–stabilizable in* H_p *steps.*

Proof. The proof of the Theorem will be given in Section 5 by showing that the $[k_p]$qdb–controller is $\big(Q_n(\Delta)\,,\,Q_n(\epsilon)\big)$–stabilizing in H_p steps. □

Theorem 2 is the main contribution of this paper. Actually, although the sufficiency of the bound is proved by exhibition of a controller achieving the desired performance, the result should be interpreted as the condition for the *existence* of a stabilizing feedback law. It is then interesting to look for other control laws different from the saturated quantized deadbeat. To this aim Theorem 2 is useful because, as it is explained below, it provides a condition for the applicability of *Model Predictive Control* techniques (MPC) which enable us to construct a family of $\big(Q_n(\Delta)\,,\,Q_n(\epsilon)\big)$–stabilizing feedback laws.

Let $\Delta > \epsilon > 0$, assume that $\mathcal{U} \subseteq \epsilon\,\mathbb{Z}$ is such that $Q_n(\epsilon)$ is controlled–invariant, hence define a feedback law

$$F_\epsilon : Q_n(\epsilon) \longrightarrow \mathcal{U}$$

rendering $Q_n(\epsilon)$ positively invariant. We use model predictive control techniques to define a controller in $Q_n(\Delta) \setminus Q_n(\epsilon)$ that steers the states to within $Q_n(\epsilon)$ in finite time, then switch to the feedback law F_ϵ.

To this aim, let $L(x,u) = I_{^cQ_n(\epsilon)}(x) \cdot \big(x'Qx + Ru^2\big)$ represent a cost function, where: $I_{^cQ_n(\epsilon)}$ is the characteristic function † of $^cQ_n(\epsilon)$, $Q \in \mathbb{R}^{n\times n}$ and $Q = Q' > 0$, $R \in \mathbb{R}$ and $R > 0$. For a fixed a number of steps $H_p > 0$, the model predictive controller is defined as

$$u(x) = U_0^*(x)\,,$$

where $U_0^*(x) \in \mathcal{U}$ is the first element of a minimizing sequence (if it exists) $U^*(x) = \big(U_0^*(x), U_1^*(x), \ldots, U_{H_p-1}^*(x)\big) \in \mathcal{U}^{H_p}$ of the following optimization problem:

$$\text{(6a)} \qquad \min_{U\in\mathcal{U}^{H_p}} \left\{ J(U,x) = \sum_{k=0}^{H_p-1} L\big(x(k), U_k\big) \right\}$$

subject to

$$\text{(6b)} \qquad \begin{cases} x(0) := x & \\ x(k+1) := Ax(k) + bU_k\,, & k = 0,\ldots,H_p-1 \\ x(k+1) \in Q_n(\Delta)\,, & k = 0,\ldots,H_p-1 \\ x(H_p) \in Q_n(\epsilon)\,, & \end{cases}$$

where $x(0)$ is the current state, $\big(x(1),\ldots,x(H_p)\big)$ is the predicted trajectory for the future H_p steps when the control sequence $U = \big(U_0, U_1, \ldots, U_{H_p-1}\big) \in \mathcal{U}^{H_p}$ is applied.

†That is $I_{^cQ_n(\epsilon)}(x) = \begin{cases} 1 & \text{if} \quad x \in {}^cQ_n(\epsilon) \\ 0 & \text{otherwise}\,. \end{cases}$.

Assume that $\forall\, x \in Q_n(\Delta)$ the optimization problem (6) is solvable (i.e., the minimum is attained), then the feedback law

$$F : Q_n(\Delta) \longrightarrow \mathcal{U}$$

$$F(x) := \begin{cases} F_\epsilon(x) & \text{if } x \in Q_n(\epsilon) \\ U_0^*(x) & \text{otherwise}\,, \end{cases} \tag{7}$$

is well defined and will be referred to as quantized–MPC controller.

Proposition 3. *The quantized–MPC controller is* $\big(Q_n(\Delta), Q_n(\epsilon)\big)$*–stabilizing.*

Proof. The proof of the Proposition follows the same arguments used for the classical dual–mode MPC scheme: see for instance [18]. □

In order to apply MPC techniques we must first guarantee that $\forall\, x \in Q_n(\Delta)$ the optimization problem (6) is solvable. To this aim it is sufficient to ensure the existence of a control sequence $U \in \mathcal{U}^{H_p}$ so that the constraints (6b) are satisfied. The quantized–MPC controller is well defined if and only if $\forall\, x \in Q_n(\Delta)$ there exists $U \in \mathcal{U}^{H_p}$ such that the predicted trajectory lies within $Q_n(\Delta)$ and enters $Q_n(\epsilon)$ after H_p steps: this is equivalent to the requirement that the system is $\big(Q_n(\Delta)\,,\, Q_n(\epsilon)\big)$–stabilizable in H_p steps. By the way, notice that even if such condition is satisfied, the feedback law (7) is $\big(Q_n(\Delta)\,,\, Q_n(\epsilon)\big)$–stabilizing but not necessarily in H_p steps since at each time instant only the first element of the minimizing sequence is applied.

When $\mathcal{U} = \epsilon\,\mathbb{Z}$ the quantized–MPC controller is well defined $\forall\, H_p \geq n$ as a consequence of Theorem 1. In the finite control set case the problem is more complicate: a sufficient condition ensuring that the quantized–MPC controller is well defined is provided by Theorem 2.

For a fixed number of steps H_p, different quantized–MPC controllers achieving the $\big(Q_n(\Delta)\,,\, Q_n(\epsilon)\big)$–stability can be obtained by varying at discretion the matrices Q and R which enter in the definition of the model predictive controller. Also H_p is a parameter that can be varied provided the optimization problem remains solvable: an obvious necessary condition is $H_p \geq n$.

The quantized-MPC controller can be implemented by modelling system (1) as a *Mixed Logical Dynamical* system (see [2, 16]): in this framework efficient algorithms to solve the optimization problem (6) are available.

For a more detailed treatment we refer to [16, 17] and to the literature about model predictive control (see in particular [12]).

4. Characterization of controlled–invariant sets

Since the aim is the stabilization of the system near the origin, we are interested in confining the trajectories within small controlled–invariant neighborhoods of 0. It is then proper to investigate the minimality properties of $Q_n(\epsilon)$. We first

review two theorems (proved in [14]) on the minimality of $Q_n(\epsilon)$ that provide the so–called *weak* and *strong* minimality properties, then we give an alternative proof of the *strong* minimality theorem which is more constructive than the one presented in [14].
Obviously, if A is a stable matrix, there exist invariant sets of arbitrarily small size: therefore we will be interested only in the case of unstable matrices.
Throughout this section we will assume without loss of generality that $\mathcal{U} \subseteq \mathbb{Z}$, that is $\epsilon = 1$.

THEOREM 3. *[Weak minimality] If Ω is a bounded controlled–invariant neighborhood of the origin and A is an unstable matrix, then $\forall\, i = 1, \ldots, n$, $diam_i\, \Omega \geq 1$.* □

The general property for controlled–invariant sets stated in Theorem 3 provides also a minimality property for $Q_n(1)$: indeed such set has the minimum diameter in all the coordinate directions. In particular, even if controlled–invariant neighborhoods of the origin contained in $Q_n(1)$ can exist, they have the same size as $Q_n(1)$. The size is measured in terms of the diameters of the set along the directions of the coordinate axes.

EXAMPLE 1. Consider $Q_n^o(1)$: it holds that $\forall\, x \in Q_n^o(1)\ \exists!\, u \in \mathbb{Z}$ such that $x^+ \in Q_n^o(1)$ (see [14]). It is hence univocally defined the mapping

$$
T: \begin{array}{ccc} Q_n^o(1) & \to & Q_n^o(1) \\ x & \mapsto & x^+ \end{array}, \tag{8}
$$

where $x^+ = Ax + b\, u(x)$ and $u(x) \in \mathbb{Z}$.
Assume that A is an unstable matrix such that $0 < |\det A| < 1$. $TQ_n^o(1)$ is obviously a controlled–invariant neighborhood of the origin. Moreover, $TQ_n^o(1)$ is strictly contained in $Q_n^o(1)$ because, denoted by λ the Lebesgue measure, from $|\mathrm{det} A| < 1$ it follows that $\lambda\big(TQ_n^o(1)\big) < \lambda\big(Q_n^o(1)\big)$.
Hence,

$$
Q_n^o(1) \supset TQ_n^o(1) \supset \cdots \supset T^k Q_n^o(1) \supset \cdots
$$

is a strictly decreasing sequence of controlled–invariant neighborhoods of the origin. The typical structure of one of the sets of the sequence (in the two dimensional case) is represented by the shaded region in Fig. 1. ♣

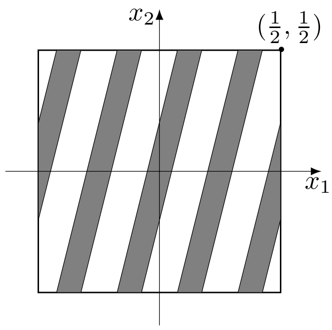

Figure 1: $TQ_n^o(1)$

THEOREM 4. *[Strong minimality] If* $|\alpha_1| > 1 + \sum_{i=2}^n |\alpha_i|$ *and* $\Omega \subseteq Q_n^o(1)$ *is a controlled–invariant neighborhood of the origin, then* $\Omega = Q_n^o(1)$ *.*

Proof. The matrix A is invertible and

$$A^{-1} = \begin{pmatrix} -\frac{\alpha_2}{\alpha_1} & -\frac{\alpha_3}{\alpha_1} & \cdots & -\frac{\alpha_n}{\alpha_1} & \frac{1}{\alpha_1} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} .$$

Hence,

$$(9) \qquad (A^{-1}x)_j = \begin{cases} \frac{x_n - \sum_{i=1}^{n-1} \alpha_{i+1} x_i}{\alpha_1} & \text{if } j = 1 \\ x_{j-1} & \text{otherwise} . \end{cases}$$

Let $\theta := \frac{(1+\sum_{i=2}^n |\alpha_i|)}{|\alpha_1|}$, by the hypothesis $\theta < 1$. $\forall\, x \in \mathbb{R}^n$,

$$(10) \qquad \left|(A^{-1}x)_1\right| \leq \frac{(1 + \sum_{i=2}^n |\alpha_i|)}{|\alpha_1|} \cdot \|x\|_\infty = \theta \cdot \|x\|_\infty < \|x\|_\infty .$$

Equations (9) and (10) imply that $A^{-1}Q_n^o(1) \subset Q_n^o(1)$, thus

$$(11) \qquad A^{-h}Q_n^o(1) \subseteq A^{-h+1}Q_n^o(1) \subseteq \cdots \subseteq A^{-1}Q_n^o(1) \subset Q_n^o(1) \quad \forall\, h \in \mathbb{N} .$$

Moreover, $A^{-n}Q_n^o(1) \subseteq Q_n(\theta)$ in fact: by Equation (9) it holds that $\forall\, x \in Q_n^o(1)$ and $\forall\, i = 1, \ldots, n$, $\big(A^{-n}x\big)_i = \big(A^{i-n-1}x\big)_1$ and, by Equations (10) and (11), $\big|(A^{i-n-1}x)_1\big| \leq \frac{\theta}{2}$. Similarly, $A^{-nk}Q_n^o(1) \subseteq Q_n(\theta^k)$ $\forall\, k \in \mathbb{N}$. Since $\lim_{k\to+\infty} \theta^k = 0$ and Ω is a neighborhood of the origin, $\exists\, k \in \mathbb{N}$ such that $Q_n(\theta^k) \subseteq \Omega$, therefore $A^{-nk}Q_n^o(1) \subseteq \Omega$.
Let $T :\ Q_n^o(1) \to Q_n^o(1)$ be the map defined in Equation (8): the controlled–invariance of Ω is equivalent to $T\,\Omega \subseteq \Omega$. We claim that $T^{nk}\big(A^{-nk}Q_n^o(1)\big) =$

$Q_n^o(1)$, the claim implies the thesis because $Q_n^o(1) = T^{nk}\big(A^{-nk} Q_n^o(1)\big) \subseteq T^{nk}\,\Omega \subseteq \Omega$.
To prove the claim, thanks to Equation (11), it is sufficient to show that $\forall\, x \in A^{-1} Q_n^o(1)$ the map T coincides with A: $Tx = Ax$ if and only if the unique control $u(x) \in \mathbb{Z}$ such that $x^+ \in Q_n^o(1)$ is $u(x) = 0$, which is the case $\forall\, x \in A^{-1} Q_n^o(1)$. $\square$

It can be shown that the condition ensuring the strong minimality of $Q_n^o(1)$ is only sufficient, nevertheless the result is interesting because it shows that there are cases in which, among the minimal diameter sets (i.e., $\operatorname{diam}_i \Omega = 1$ $\forall\, i = 1, \dots, n$), the whole $Q_n^o(1)$ is actually the minimal one.

5. Proof of Theorem 2

Theorem 2 was first stated without proof in [15]. Although the proof appears to be complicate, it is instead based on simple ideas trickly exploiting the properties of the controller form coordinates. Hence, it is worth recalling that the control acts only on the n^{th} component while the others shift upward.
To prove the theorem we will take advantage of some lemmas: the hypotheses of Theorem 2 are implicitly assumed.

Lemma 2. *The sequence* $\{\psi_m\}_{m \in \mathbb{N}\setminus\{0\}}$ *(see Equation (5)) is non–decreasing.*

Proof. We argue by induction: $\psi_1 = 1$, $\psi_2 = 1 + |\alpha_n| \geq \psi_1$.
Assume that $\psi_h \leq \psi_{h+1}$ $\forall\, h < m$, then $\psi_m \leq \psi_{m+1}$. In fact: $\psi_{m+1} - \psi_m = \sum_{i=1}^{m} |\alpha_{n-m+i}|\psi_i - \sum_{i=1}^{m-1} |\alpha_{n-m+i+1}|\psi_i = |\alpha_{n-m+1}|\psi_1 + \sum_{i=2}^{m} |\alpha_{n-m+i}|(\psi_i - \psi_{i-1}) \geq 0$ by the inductive hypothesis. $\square$

We extend the sequence $\{\psi_m\}_{m \in \mathbb{N}\setminus\{0\}}$ defining $\psi_z = 0$ $\ \forall\, z \in \mathbb{Z}$, $z \leq 0$.

Let

$$\varphi := \frac{\Delta}{2}\Big(1 - \sum_{i=1}^{n} |\alpha_i|\Big) + k_p \epsilon\,. \tag{12}$$

By the definition of k_p (see Equation (4)) and the fact that $\lceil x \rceil = x + \theta$ $(0 \leq \theta < 1)$, it is easy to check that $\varphi > 0$.

Lemma 3. *Suppose* A *is such that* $\alpha_i \geq 0$ $\ \forall\, i = 1, \dots, n$. *Consider the sequence* $\{x(t)\}_{t \in \mathbb{T} \subseteq \mathbb{N}}$ *recursively defined by*

$$\begin{cases} x(0) := (\frac{\Delta}{2}, \dots, \frac{\Delta}{2}) \\ \textit{while } \big(x_n(t) > \frac{\epsilon}{2}\big) \quad \textit{let} \quad x(t+1) := Ax(t) - b \cdot (k_p \epsilon)\,; \end{cases}$$

then $x_i(t) = \frac{\Delta}{2} - \varphi\,\psi_{t-n+i}$ $\ \forall\, i = 1, \dots, n$ *and* $\forall\, t \in \mathbb{T}$.
Moreover, if $x_n(t) > \frac{\epsilon}{2}$ *then* $x_i(t) > 0$ $\ \forall\, i = 1, \dots, n$.

Proof. By induction: when $t=0$ the statement is obvious.
Suppose that $x_n(t) > \frac{\epsilon}{2}$ and that $x_i(t) = \frac{\Delta}{2} - \varphi\,\psi_{t-n+i}\ \ \forall\, i = 1,\ldots,n$, then $x_i(t+1) = \frac{\Delta}{2} - \varphi\,\psi_{t+1-n+i}$. In fact:
if $i < n$, since A is in controller form, $x_i(t+1) = x_{i+1}(t) = \frac{\Delta}{2} - \varphi\,\psi_{t-n+i+1}$.
When $i = n$, using respectively Equation (12), $\psi_z = 0$ for $z \le 0$ and Equation (5), we have: $x_n(t+1) = \sum_{l=1}^n \alpha_l x_l(t) - k_p\epsilon = \sum_{l=1}^n \alpha_l(\frac{\Delta}{2} - \varphi\,\psi_{t-n+l}) + \frac{\Delta}{2} - \frac{\Delta}{2}\sum_{i=l}^n \alpha_l - \varphi = \frac{\Delta}{2} - \varphi(1 + \sum_{l=1}^n \alpha_l\psi_{t-n+l}) = \frac{\Delta}{2} - \varphi(1 + \sum_{l=n+1-t}^n \alpha_l\psi_{t-n+l}) = \frac{\Delta}{2} - \varphi(1 + \sum_{j=1}^t \alpha_{j-t+n}\psi_j) = \frac{\Delta}{2} - \varphi\,\psi_{t+1}$.
The statement $x_n(t) > \frac{\epsilon}{2} \ \Rightarrow\ x_i(t) > 0$ follows immediately by the definition of the sequence $\{x(t)\}_{t\in\mathbb{T}}$ and the controller form of A. □

Denote by $\Xi(|A|, k_p)$ the saturation region of the $[k_p]$qdb–controller for system $(|A|, b)$. We say that $x \in \mathbb{R}^n$ satisfies the property (P_{k_p}) iff:

$$\begin{cases} x_i \ge 0 \quad \forall\, i = 1,\ldots,n \\ x \notin \Xi(|A|, k_p)\,. \end{cases} \tag{P_{k_p}}$$

In the proof of Theorem 2 we shall make extensive use of the following

Lemma 4. *If x satisfies the property (P_{k_p}) and y is such that $|y_i| \le x_i\ \forall\, i = 1,\ldots,n$, then the closed–loop dynamics induced by the $[k_p]$qdb–controller for system (A,b) is such that $|y_n^+| \le \frac{\epsilon}{2}$.*

Proof. We show that $\big|\sum_{i=1}^n \alpha_i y_i\big| \le \frac{\epsilon}{2} + k_p\epsilon$ and conclude applying Lemma 1:
$\big|\sum_{i=1}^n \alpha_i y_i\big| \le \sum_{i=1}^n |\alpha_i| x_i \le \frac{\epsilon}{2} + k_p\epsilon$ because $x \notin \Xi(|A|, k_p)$ and Lemma 1. □

Proof of Theorem 2. We show that the $[k_p]$qdb–controller is $\big(Q_n(\Delta)\,,\, Q_n(\epsilon)\big)$–stabilizing in $H_p = n + p - 1$ steps.
Let $k(\Delta)$ be as in Equation (3), since $k_p \ge k(\Delta) \ge k(\epsilon)$ the positive invariance of $Q_n(\Delta)$ and $Q_n(\epsilon)$ is ensured by Proposition 1. The proof of the convergence to $Q_n(\epsilon)$ in the desired number of steps is organized as follows: we first suppose that $\alpha_i \ge 0\ \forall\, i = 1,\ldots,n$ and prove that the property holds for the trajectory starting from $x(0) := (\frac{\Delta}{2}, \ldots, \frac{\Delta}{2})$. This is obtained by showing that:
Statement I) Let $\tilde{t}$ be such that $|x_n(\tilde{t})| \le \frac{\epsilon}{2}$ and $|x_n(t)| > \frac{\epsilon}{2}\ \ \forall\, t < \tilde{t}$, then $|x_n(t')| \le \frac{\epsilon}{2}\ \ \forall\, t' \ge \tilde{t}$;
Statement II) $\tilde{t} \le p$.
Statements I+II imply the assertion for $x(0)$ because $|x_n(p+h)| \le \frac{\epsilon}{2}\ \ \forall\, h \ge 0$ and A is in controller form.
The general case $\big(y(0) \in Q_n(\Delta)$ and arbitrary α_i's$\big)$ is proved by comparing the trajectories of the system with the one analyzed in the first part of the proof.

• *First case:* $\alpha_i \geq 0 \ \ \forall\, i = 1, \ldots, n\,, \quad x(0) = (\frac{\Delta}{2}, \ldots, \frac{\Delta}{2})\,.$

Proof of statement I) For $t < \tilde{t}$, the state evolves according to the sequence defined in Lemma 3 , therefore

$$x_i(t) > 0 \qquad \forall\, t < \tilde{t} \ \text{ and } \ \forall\, i = 1, \ldots, n\,. \tag{13}$$

Equation (13) and $|x_n(\tilde{t})| \leq \frac{\epsilon}{2}$ imply that $x(\tilde{t}-1)$ satisfies the property (P_{k_p}). By Lemma 3 we know that $x_i(\tilde{t}-1) = \frac{\Delta}{2} - \varphi\, \psi_{\tilde{t}-1-n+i} \ \ \forall\, i = 1, \ldots, n\,.$
We show by induction that

$$\forall\, h \geq 0\,, \quad \begin{cases} |x_i(\tilde{t}+h)| \leq x_i(\tilde{t}-1) & \forall\, i = 1, \ldots, n \\ |x_n(\tilde{t}+h)| \leq \dfrac{\epsilon}{2}\,; \end{cases}$$

this proves statement I .
Case $h = 0$:
if $i < n$ then $|x_i(\tilde{t})| = x_{i+1}(\tilde{t}-1) = \frac{\Delta}{2} - \varphi\, \psi_{\tilde{t}-n+i} \leq \frac{\Delta}{2} - \varphi\, \psi_{\tilde{t}-n+i-1} = x_i(\tilde{t}-1)\,,$ where the inequality follows by Lemma 2 .
If $i = n$ then $|x_n(\tilde{t})| \leq \frac{\epsilon}{2} < x_n(\tilde{t}-1)\,.$
Inductive step $h \Rightarrow h+1$:
if $i < n$ then $|x_i(\tilde{t}+h+1)| = |x_{i+1}(\tilde{t}+h)| \leq x_{i+1}(\tilde{t}-1)$ by the inductive hypothesis; $x_{i+1}(\tilde{t}-1) \leq x_i(\tilde{t}-1)$ as shown in case $h = 0$.
If $i = n$ we know by the inductive hypothesis that $|x_j(\tilde{t}+h)| \leq x_j(\tilde{t}-1)$ $\forall\, j = 1, \ldots, n$, hence by Lemma 4 it follows that $|x_n(\tilde{t}+h+1)| \leq \frac{\epsilon}{2} < x_n(\tilde{t}-1)\,.$

Proof of statement II) Because of statement I it is sufficient to show that $|x_n(p)| \leq \frac{\epsilon}{2}$. Suppose that $x_n(t) > \frac{\epsilon}{2} \quad \forall\, t < p$ $\big($we have dropped the modulus because of Equation (13)$\big)$, let $r := \sum_{i=1}^{n} \alpha_i x_i(p-1) - k_p \epsilon = \frac{\Delta}{2} - \varphi\, \psi_p$ by Lemma 3. By Lemma 1 $|x_n(p)| \leq \frac{\epsilon}{2}$ if and only if $r \leq \frac{\epsilon}{2}$, namely:

$$\frac{\Delta}{2} - \varphi\, \psi_p \leq \frac{\epsilon}{2} \ \Leftrightarrow\ \varphi \geq \frac{\Delta - \epsilon}{2\psi_p}\,.$$

By Equation (12) $\varphi = \frac{\Delta}{2} - \frac{\Delta}{2} \sum_{i=1}^{n} \alpha_i + k_p \epsilon$, it is then sufficient to show that

$$\min \left\{ m \in \mathbb{N} \,\Big|\, \frac{\Delta}{2} - \frac{\Delta}{2} \sum_{i=1}^{n} \alpha_i + m\epsilon \geq \frac{\Delta - \epsilon}{2\psi_p} \right\} = k_p\,. \tag{14}$$

Indeed, solving for $\mu \in \mathbb{R}$:

$$\frac{\Delta}{2} - \frac{\Delta}{2} \sum_{i=1}^{n} \alpha_i + \mu\epsilon \geq \frac{\Delta - \epsilon}{2\psi_p} \iff \mu \geq \frac{1}{2} \frac{\Delta}{\epsilon} \Big(\sum_{i=1}^{n} \alpha_i - 1 \Big) + \frac{1}{\epsilon} \frac{\Delta - \epsilon}{2\psi_p} := \mu_{\min}\,,$$

hence the integer minimum in Equation (14) is $\lceil \mu_{\min} \rceil = k_p$. This concludes the first part of the proof.

From the discussion above it follows immediately that if $\mathcal{U} = \mathcal{U}_{k_p-1}$ and $\alpha_i \geq 0 \ \ \forall\, i = 1, \ldots, n$, then the system is not $\big(Q_n(\Delta), Q_n(\epsilon)\big)$–stabilizable in H_p steps.

• *General case: arbitrary α_i's and $y(0) \in Q_n(\Delta)$.*

The thesis is obtained by comparing the evolution of $y(0)$ according to the $[k_p]$qdb–controller and the evolution of $x(0) = (\frac{\Delta}{2}, \dots, \frac{\Delta}{2})$ driven by system $(|A|, b)$ and the corresponding $[k_p]$qdb–controller.
From the first part of the proof we know that $\exists\, \tilde{t} \le p$ such that $|x_n(\tilde{t})| \le \frac{\epsilon}{2}$ and $x_n(t) > \frac{\epsilon}{2} \quad \forall\, t < \tilde{t}$. Moreover, property (P_{k_p}) holds for $x(\tilde{t}-1)$.
First we show by induction that $\forall\, h \le \tilde{t}-1$ and $\forall\, i = 1, \dots, n$, $|y_i(h)| \le x_i(h)$: the case $h = 0$ is obvious.
If $h < \tilde{t} - 1$ let us show the inductive step $h \Rightarrow h+1$:
if $i < n$ then $|y_i(h+1)| = |y_{i+1}(h)| \le x_{i+1}(h)$ because of the inductive hypothesis; also, $x_{i+1}(h) = x_i(h+1)$.
For $i = n$, if $|y_n(h+1)| \le \frac{\epsilon}{2}$ then $|y_n(h+1)| \le \frac{\epsilon}{2} < x_n(h+1)$ because $h + 1 \le \tilde{t} - 1$.
If instead $|y_n(h+1)| > \frac{\epsilon}{2}$, by Lemma 1 and the definition of the $[k_p]$qdb–controller, $y_n(h+1) = \sum_{j=1}^n \alpha_j y_j(h) \pm k_p\epsilon$ (with $+$ if $\sum_{j=1}^n \alpha_j y_j(h) < 0$ and vice versa). Since $h+1 \le \tilde{t}-1$, then $x_n(h+1) = \sum_{j=1}^n |\alpha_j| x_j(h) - k_p\epsilon$. Let us suppose that $\sum_{j=1}^n \alpha_j y_j(h) > 0$ (the opposite case is analogue): $y_n(h+1) = \sum_{j=1}^n \alpha_j y_j(h) - k_p\epsilon > \frac{\epsilon}{2}$, thus $|y_n(h+1)| = y_n(h+1) \le \sum_{j=1}^n |\alpha_j||y_j(h)| - k_p\epsilon \le \sum_{j=1}^n |\alpha_j| x_j(h) - k_p\epsilon = x_n(h+1)$ where the last inequality follows by the inductive hypothesis.
In particular $|y_i(\tilde{t}-1)| \le x_i(\tilde{t}-1) \quad \forall\, i = 1, \dots, n$: hence, as property (P_{k_p}) is satisfied by $x(\tilde{t}-1)$, by Lemma 4 it holds that $|y_n(\tilde{t})| \le \frac{\epsilon}{2}$. Since $\tilde{t} \le p$, to conclude the proof it is sufficient to show that

$$\forall\, h \ge 0\,, \quad \begin{cases} |y_i(\tilde{t}+h)| \le x_i(\tilde{t}-1) & \forall\, i = 1, \dots, n \\ |y_n(\tilde{t}+h)| \le \dfrac{\epsilon}{2}\,. \end{cases}$$

We prove it by induction. Case $h = 0$:
if $i < n$ then $|y_i(\tilde{t})| = |y_{i+1}(\tilde{t}-1)| \le x_{i+1}(\tilde{t}-1)$ as proved above. We already know (see the proof of statement I) that $x_{i+1}(\tilde{t}-1) \le x_i(\tilde{t}-1)$.
If $i = n$ then $|y_n(\tilde{t})| \le \frac{\epsilon}{2} < x_n(\tilde{t}-1)$.
The inductive step $h \Rightarrow h+1$ can be proved in the same way as the analogue property showed in the proof of statement I. □

6. Conclusion

We have considered the practical stabilization problem for discrete–time linear systems subject to a fixed uniformly quantized control set. Several results have been derived taking advantage of the controller form coordinates. In particular we have provided results on the existence and construction of controlled–invariant sets (including a general characterization of such sets) and for the synthesis of stabilizing control laws. The approach is promising also to solve more general problems in the most important and challenging area where quantization is combined with limited communication bandwidth.

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