Discrete and Hybrid Nonholonomy*

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Abstract. In this paper we consider the generalization of the classical notion of nonholonomy of smooth constraints in analytical mechanics, to a substantially wider set of systems, allowing for discrete and hybrid (mixed continuous and discrete) configurations and transitions. We show that the general notion of nonholonomy can be captured by the definition of two different types of nonholonomic behaviours, which we call *internal* and *external*, respectively. Examples are reported of systems exhibiting either the former only, or the latter only, or both. For some classes of systems, we provide equivalent or sufficient characterizations of such definitions, which allow for practical tests.

1 Introduction

Although nonholonomic mechanics has a long history, dating back at least to the work of Hertz and Hölder towards the end of the 19th century, it is still today a very active domain of research, both for its theoretical interest and its applications, e.g. in wheeled vehicles, robotics, and motion generation. In the past decade or so, a flurry of activity has concerned the study of nonholonomic systems as nonlinear dynamic systems to which control theory methods could be profitably applied. As a result, the control of classical nonholonomic mechanical systems such as cars, trucks with trailers, rolling 3D objects, underactuated mechanisms, satellites, etc., has made a definite progress, and often met a satisfactory level.

Systems considered in classical nonholonomic mechanics are smooth, continuous time systems, i.e., they can be described by ODEs on a smooth manifold of configurations, on which smooth (most often, analytic) constraints apply. However, nonholonomic-like behaviours can be recognized in more general systems, some of great practical relevance, which may present for instance discontinuities of the dynamics, discreteness of the time axis, and discreteness (e.g., quantization) of the input space. For these systems, some very basic control problems such as the analysis of reachability and the synthesis of steering control sequences still pose quite challenging problems.

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This paper attempts at providing a general conceptual framework capable of capturing the notion of nonholonomy for a broad class of systems, allowing for discrete and hybrid (mixed continuous and discrete) configurations and transitions. Upon the analysis of few simple but significant examples, a unique definition encompassing all "intuitively nonholonomic" behaviours in hybrid systems, does not appear to be feasible, or practical. Hence we propose the definition of two different types of nonholonomic behaviours, which we call *internal* and *external*, respectively. These two types are not obviously reducible to a single one, and indeed we show examples of simple mechanical systems exhibiting only internal, only external, or both internal and external nonholonomy, respectively. Although our definitions are not always directly computable, we provide equivalent, or sufficient conditions for some specific classes of systems, which allow for practical tests to be applied.

2 Nonholonomic behaviours in nonsmooth systems

In general, classical nonholonomic constraints come in two varieties, kinematic constraints (often due to contact kinematics, as e.g. in rolling), and dynamic constraints (due to symmetries induced by conservation laws, for instance, of angular momentum) [1, 2]. In this paper we focus on the former type. Recall the definition of a (smooth) nonholonomic constraint that is familiar from elementary mechanics textbooks: a mechanical system described by coordinates $q \in \mathcal{Q}$, with $\mathcal{Q}$ a smooth n-dimensional manifold, subject to m smooth constraints $A(q)\dot{q} = 0$, is nonholonomic if $A(\cdot)$ is not integrable.

An equivalent description of such systems is often useful, which uses a basis $G(q)$ of the distribution that annihilates $A(q)$ to describe allowable velocities $\dot{q} \in T_q\mathcal{Q}$ as

$$\dot{q} = G(q)u. \tag{1}$$

Thanks to Frobenius' theorem, nonholonomy can thus be investigated by studying the Lie algebra generated by the vector fields in $G(q)$, or, in other terms, by analyzing the geometry of the reachability set of (1). Such simple formulation of kinematic nonholonomic systems is sufficient to illustrate two fundamental aspects of nonholonomy:

1) elements of $u \in \mathbb{R}^{n-m}$ in (1) play the role of control inputs in a nonlinear, affine–in–control, driftless dynamic system. If the original constraint is nonholonomic, the dimension of the reachable manifold is larger than the number of inputs. This has motivated purposeful introduction of nonholonomy in the design of mechanical devices, to spare actuator hardware while maintaining steerability (see e.g. [3, 4]). Notice explicitly that for driftless systems, reachability on a manifold with dimension larger than the dimension of the input space is an essentially nonlinear phenomenon, which is altogether destroyed by linearization, and can be considered as a synonim of nonholonomy;

2) the effects of different consecutive inputs in nonholonomic systems do not commute. Moreover, such noncommutative inputs may produce net motions of the system in directions not belonging to the input distribution evaluated at the starting point. This observation is crucial in the interpretation of the role of Lie–brackets in deciding integrability of the system[5].

Behaviors that, by similarity, could well be termed “nonholonomic”, may actually occur in a much wider class of systems than mechanical systems with smooth contact constraints or symmetries. Let us refer to general time-invariant dynamic systems as a quintuple $\Sigma = (\mathcal{Q}, \mathcal{U}, \Omega, \mathcal{A})$, with $\mathcal{Q}$ denoting the configuration set, $\mathcal{U}$ a set of admissible input symbols, Ω a set of admissible input streams (continuous functions, or discrete sequences) formed by symbols in $\mathcal{U}$, and $\mathcal{A}$ a state–transition map $\mathcal{A} : \mathcal{Q} \times \Omega \to \mathcal{Q}$. In many cases $\mathcal{U}$ is determined by a set of controls defined on an ordered time set, $\mathcal{T}$.

It has been observed that in piecewise smooth (p.s.) systems (where time is continuous, $\mathcal{Q}$ is a p.s. manifold, and $\mathcal{A}$ is a p.s. map) with holonomic dynamics within each smooth region, nonholonomic behaviours can be introduced by switching among different smooth regions of the configuration space. Piecewise holonomic systems have been studied rather extensively (see e.g. [6–10]). A prominent role in the study of p.s. nonholonomic systems is played by tools from differential geometric control theory (cf. [1, 2]) and from the theory of stratified manifolds ([11]).

Nonholonomic behaviors may also be exhibited by discrete–time systems ($\mathcal{T} = \mathbb{N}$). Consider that, if $\mathcal{Q}$ and $\mathcal{U}$ in the system quintuple represent continuous sets, a classical discrete–time control system is described. For such systems, the reachability problem has been already clarified in the literature (see e.g. [12–15]). On the other hand, if $\mathcal{Q}$ and $\mathcal{U}$ are assumed to be discrete sets, then the system essentially represents a sequential machine (automaton). Reachability questions for such systems are fundamentally equivalent to graph connectivity analysis, an extensively studied topic.

A particularly stimulating problem arises when $\mathcal{Q}$ has the cardinality of a continuum, but $\mathcal{U}$ is quantized (i.e. finite, or discrete with values on a regular mesh). Such systems, which will be referred to as quantized control systems (QCS), are encountered in many applications, due e.g. to the need of using finite–capacity digital channels to convey information through an embedded control loop, or to abstract symbolic information from too complex sensorial sources (such as video images in visual servoing applications). As a consequence, several researchers devoted their attention to this type of systems (see e.g. [16, 17, 6, 18]). It is important to notice that, while inputs are quantized, the system configurations are not a priori restricted to any finite or discrete set: thus, it may happen that the reachable set has accumulation points, or is dense in the whole space, or in some subsets, or nowhere ([19]).

Chitour and Piccoli [20] have studied a quantized control synthesis problem for the linear case $x^+ = Ax + Bu$, providing sufficient conditions and a constructive technique to find a finite input set $\mathcal{U}$ to achieve a reachability set which is dense in $\mathcal{Q}$. The analysis of the reachability set of a QCS with a given quantized input set $\mathcal{U}$, has been considered in [21, 19]. In these papers, a complete analysis is achieved for driftless linear systems (while it is pointed out that the problem for general linear systems is as though as some reputedly hard problems in number theory), and for a particular class of driftless nonlinear systems, namely the exact sampled models of n-dimensional chained–form systems ([22]), which can be considered as the simplest nonholonomic system model.

3 A set of examples

To motivate and drive our discussion, we start by illustrating few basic examples of systems whose behaviour we should like any definition of hybrid nonholonomy to be able to capture.

Example 1. A first set of elementary examples is obtained by considering the Heisenberg-Brockett nonholonomic integrator ([6])

$$Dq = \begin{bmatrix} 1 \\ 0 \\ -y \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} u_2, \quad q \in \mathcal{Q} = \mathbb{R}^3, \tag{2}$$

in four different settings:

1-i) Continuous time ($t \in \mathcal{T} = \mathbb{R}^+$, $Dq := \frac{d}{dt}q(t)$), continuous control ($u \in \mathcal{U} = \mathbb{R}^2$). The system is nonholonomic in the classical sense.

1-ii) Discrete time ($t \in \mathcal{T} = \mathbb{N}$, $Dq := q(t+1) - q(t)$), continuous control;

1-iii) Continuous time, quantized control ($u \in \mathcal{U}$, Card $(\mathcal{U}) \in \mathbb{N}$, $\Omega =$ piecewise-constant functions with values in $\mathcal{U}$. For instance, take $\mathcal{U} = \{(u_1, u_2)^T | \; u_1 \in \{0, a, -a\}, \; u_2 \in \{0, b, -b\}\}$, for some constant $a, b \in \mathbb{R}$;

1-iv) Discrete time, quantized control.

Example 2. As an example of a piecewise holonomic system, we will consider the simplified version of one of Brockett's rectifiers ([23]) in figure 1. The tip of

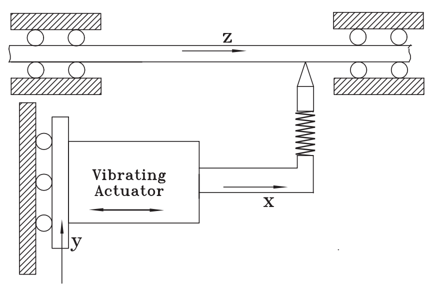

Fig. 1. A micro-electro-mechanical (M.E.M.) motion rectifier illustrating the definition of external nonholonomy in a piecewise holonomic system.

a piezoelectric or electrostrictive element oscillates in the x–direction, while an actuator drives the oscillator support along the y–direction. When y reaches a threshold y_0, dry friction is sufficient to push the rod in the z–direction. Disregarding dynamics, the rectifier can be modeled by a continuous–time system with configurations $q = (x, y, z) \in \mathcal{Q} = \mathbb{R}^3$. Assuming that the velocity of the support ($\dot{y}$), and of the oscillator tip ($\dot{x}$) can be freely chosen, a model for this system congruent with the definitions above would be

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_3$$

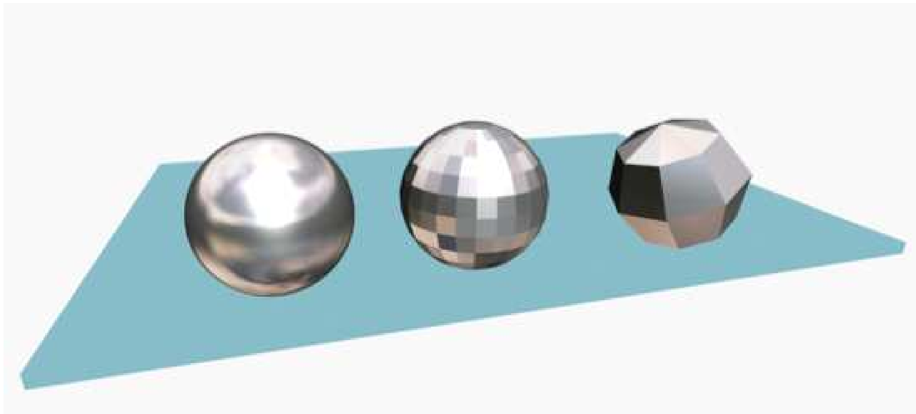

Fig. 2. Three discrete approximations of the plate-ball systems.

with the input restrictions

$$\begin{cases} u_3 = 0 \; y < y_0 \\ u_2 = 0 \; y \geq y_0 \end{cases} .$$

Example 3. As a third example, we consider a system comprised of a polyhedron with one face lying on a plane, which is rolled by control actions which place one of the adjacent faces on the plane (i.e., by rotating the polyhedron about one of the edges of the face currently in contact by the exact amount that brings an adjacent face onto the plane). This can be regarded as a discrete approximation of the plate-ball system (see fig. 2), a standard example in nonholonomic textbooks. Although it may seem intuitive that "nonholonomy" is conserved by at least the finest approximations, no current definition of "nonholonomy" would be applicable to this example.

4 Discrete Nonholonomy

From consideration of examples 2 and 3, it follows directly that to afford the generality we aim at, the input set in the system quintuple Σ should be state–dependent. In other words, different sets of input actions may be available at different states, as it is clearly the case for the polyhedron when lying with different faces on the plane. To deal with this problem, let us be more specific on the definition of the input set $\mathcal{U}$, and assume that there exists a multivalued function $\phi : \mathcal{Q} \to \mathcal{U}$ where $\phi(q) = \mathcal{U}_q \subset \mathcal{U}$ is the set of admissible inputs at q. Consider an input equivalence relation on $\mathcal{Q}$ given by $q_1 \overset{\mathcal{U}}{\equiv} q_2$ iff $\phi(q_1) = \phi(q_2)$, and denote $\mathcal{Q}/\phi$ the set of input equivalence classes, $[q]$ the input equivalence class of q.

Further, let Ω_q be the language over $\mathcal{U}$ consisting of admissible input streams for the system being currently in configuration q. For each $q \in \mathcal{Q}$ and $\omega \in \Omega_q$, let the end-point map, i.e. the state that the system reaches from q under $\omega \in \Omega_q$, be denoted as $\mathcal{A}(q,\cdot) : \Omega_q \to \mathcal{Q}$, or simply as $\mathcal{A}_q(\omega)$.

Two configurations q_1, q_2 are stream equivalent (denoted $q_1 \stackrel{\Omega}{\equiv} q_2$) iff $\Omega_{q_1} = \Omega_{q_2}$. Accordingly, $\mathcal{Q}/\Omega$ denotes the set of stream equivalence classes, and $[q]_\Omega$ is the stream equivalence class of q. Clearly, input and stream equivalence classes coincide if the following compatibility condition of the map $\mathcal{A}$ with the equivalence relation $\stackrel{\mathcal{U}}{\equiv}$ holds (see [24]):

[H1] $\forall q_1 \stackrel{\mathcal{U}}{\equiv} q_2$ and $\forall u \in \mathcal{U}_{q_1} (= \mathcal{U}_{q_2})$, $\mathcal{A}_{q_1}(u) \stackrel{\mathcal{U}}{\equiv} \mathcal{A}_{q_2}(u)$.

We assume in the following that $\mathcal{Q}$ is a manifold and that each input and stream equivalence classes are connected submanifolds of $\mathcal{Q}$.

Denote by $\widetilde{\Omega}_q = \{\omega \in \Omega_q : \ \mathcal{A}_q(\omega) \in [q]\}$ the sublanguage consisting of those input streams which steer the system eventually back to the same equivalence class of the initial point. For $\omega_1, \omega_2 \in \widetilde{\Omega}_q$, the stream concatenation $\omega_1\omega_2$ is well defined. The notion of kinematic (i.e., driftless) systems of the form (1) can be extended in this context by the assumption that $\widetilde{\Omega}_q$ contains an identity element, $0 \in \widetilde{\Omega}_q$, such that $\mathcal{A}_q(0) = q$, for all $q \in [q]$. In general, the language $\widetilde{\Omega_q}$ is not prefix-closed. However, we will also consider the *orbit* of $q \in [q]$ under $\widetilde{\Omega}_q$ (denoted as $\mathcal{R}_q(\overline{\widetilde{\Omega}}_q)$) as the reachable set from q under words in the prefix-closure $\overline{\widetilde{\Omega}}_q$ of $\widetilde{\Omega}_q$, in other words $\mathcal{R}_q(\overline{\widetilde{\Omega}}_q) := \left\{ p \in \mathcal{Q} : p = \mathcal{A}_q(\omega_s), \omega_s \in \overline{\widetilde{\Omega}}_q \right\}$ with $\overline{\widetilde{\Omega}}_q := \left\{ \omega_s \in \mathcal{U}^* : \exists \omega_t \in \mathcal{U}^*, (\omega_s\omega_t \in \widetilde{\Omega}_q) \right\}$.

Consideration of the examples above, and the introduction of input equivalence classes and orbits, induces us to consider two different types of behaviours which may be termed "nonholonomic" by analogy with observations made in paragraph 2 about the increased reachability afforded by cyclic controls. Loosely speaking, we will refer to the case where cyclic switchings that temporarily "get out" of an equivalence class add to reachability more than what availed by paths "staying in", as to an "external" type of nonholonomy. On the other hand, when there exist reachability-generating cycles which keep the configuration always within the same equivalence class, or orbit, then we will speak of an "internal" type of nonholonomy.

4.1 External Nonholonomy

More precisely, consider the maximal sublanguage $\widehat{\Omega}_q \subseteq \widetilde{\Omega}_q$ of words that always keep the configuration within the same equivalence class, and compare the corresponding orbit $\mathcal{R}_q(\overline{\widehat{\Omega}}_q) = \mathcal{R}_q(\widehat{\Omega}_q) \subseteq [q]$ with the set reachable from q under $\widetilde{\Omega}_q$, $\mathcal{R}_q(\widetilde{\Omega}_q) = \{\mathcal{A}_q(\omega) : \ \omega \in \widetilde{\Omega}_q\}$.

Definition 1. *A system* $(\mathcal{Q}, \mathcal{U}, \Omega, \mathcal{A})$ *is said to be externally nonholonomic at* $q \in \mathcal{Q}$ *if* $\mathcal{R}_q(\widetilde{\Omega}_q) \supsetneqq \mathcal{R}_q(\widehat{\Omega}_q)$.

Checking for external nonholonomy directly from its definitions is clearly not feasible in general. However, under some mild conditions, we can replace the set comparison in the definition with a comparison of groups, which can be easily computed in many cases, for instance comparing sets of generators for the groups themselves.

Let $\mathcal{Q}^{\mathcal{Q}}$ be the set of mappings of $\mathcal{Q}$ into itself. The action of words in Ω on $\mathcal{Q}$ $a : \Omega \to \mathcal{Q}^{\mathcal{Q}}$, $a(\omega) \mapsto \mathcal{A}(\cdot, \omega)$, with a null element $\varepsilon \in \Omega$ such that $a(\varepsilon) = \mathcal{A}(\cdot, \varepsilon) = Id$, and with the natural composition law on $\mathcal{Q}^{\mathcal{Q}}$, is a monoid homomorphism. Let $\widetilde{\mathcal{S}} \subset \mathcal{Q}^{\mathcal{Q}}$ be the subset of bijective, hence invertible, maps of $\mathcal{Q}$ into itself. Then $\widetilde{\mathcal{S}}$ is a group for the composition operation. Under the further assumption that the system is *invertible*, i.e. that

[(H2)] $\forall q \in \mathcal{Q}$, $a(\widetilde{\Omega}_q) \subset \widetilde{\mathcal{S}}$ and $\forall \omega \in \widetilde{\Omega}_q$, $\exists\, \bar{\omega} \in \widetilde{\Omega}_q$ such that $a(\omega) = (a(\bar{\omega}))^{-1}$,

we have that $\widetilde{\Omega}_q$ and $\widehat{\Omega}_q$ can be both endowed with a group structure. Under this hypothesis, we write $\omega\bar{\omega} = \bar{\omega}\omega = \varepsilon$, so that $\mathcal{A}_q(\omega\bar{\omega}) = \mathcal{A}_q(\varepsilon) = q$. Hence $\widetilde{\Omega}_q$ (or the quotient of $\widetilde{\Omega}_q$ over the corresponding equivalence relation among multiple possible inverses), is a group. We therefore have that $a : \widetilde{\Omega}_q \to \widetilde{\mathcal{S}}$ is a group homomorphism, and the following holds:

Proposition 1. *If a system $(\mathcal{Q}, \mathcal{U}, \Omega, \mathcal{A})$ is externally nonholonomic at $q \in \mathcal{Q}$ then $a(\widehat{\Omega}_q) \subsetneqq a(\widetilde{\Omega}_q)$, where the inclusion is a group inclusion.*

Notice that the converse of proposition 1 does not hold in general, as shown in this example, where $a(\widehat{\Omega}_q) \subsetneqq a(\widetilde{\Omega}_q)$ but $\mathcal{R}(\widetilde{\Omega}_q) = \mathcal{R}(\widehat{\Omega}_q)$:

Example 4 Consider a quantized system defined on $\mathbb{R}^2$ by the following control sets: $\mathcal{U}_1 = \{e, u, \bar{u}, v\}$ for $q \in \mathcal{Q}_1 = \{(q_1, q_2) : q_1 \in \mathbb{R},\ -1 < q_2 \leq 0\}$ and $\mathcal{U}_2 = \{e, w, \bar{w}, \bar{v}\}$, for $q \in \mathcal{Q}_2 = \{(q_1, q_2) : q_1 \in \mathbb{R},\ 0 < q_2 \leq 1\}$, with $\mathcal{A}_{(q_1,q_2)}(\varepsilon) = (q_1, q_2)$, $\mathcal{A}_{(q_1,q_2)}(u) = (q_1 + 1, q_2)$, $\mathcal{A}_{(q_1,q_2)}(\bar{u}) = (q_1 - 1, q_2)$, $\mathcal{A}_{(q_1,q_2)}(v) = (q_1, q_1 + 1)$, $\mathcal{A}_{(q_1,q_2)}(\bar{v}) = (q_1, q_2 - 1)$, $\mathcal{A}_{(q_1,q_2)}(w) = (-q_1, q_2) = \mathcal{A}_{(q_1,q_2)}(\bar{w})$. Set $q = (0, -1/2)$, then $\widehat{\Omega}_q = \{u^{k_1}\bar{u}^{k_2},\ k_1, k_2 \in \mathbb{Z}\}$ and $\widetilde{\Omega}_q = \widehat{\Omega}_1 \cup \{u^{k_1} v w^{k_2} \bar{v} \bar{u}^{k_3},\quad k_1, k_2, k_3 \in \mathbb{Z}\}$, hence $a(\widehat{\Omega}_q) \subsetneqq a(\widetilde{\Omega}_q)$, but $\mathcal{R}(\widetilde{\Omega}_q) = \mathcal{R}(\widehat{\Omega}_q) = \{q + (k, 0),\ k \in \mathbb{Z}\}$. The above holds true also for the choice $q = (k, \alpha)$, $k \in \mathbb{Z}$ and $-1 < \alpha \leq 0$.

However, the following holds:

Proposition 2. *Assume that on $[q]$ it is defined an operation "$\cdot$", so that $([q], \cdot)$ is a group, and "$\cdot$" is compatible with the action of $\widetilde{\Omega}_q$, in the sense that $a(\widetilde{\Omega}_q) \subset \{\varphi : [q] \mapsto [q] :\ \exists q_1 \in [q]\, s.t. \forall q_2 \in [q], \varphi(q_2) = q_1 \cdot q_2\}$. Assume also that the empty word ε in $\widetilde{\Omega}$ is the unique element of the isotropy group at q, i.e. $\{\omega \in \widetilde{\Omega}_q :\ \mathcal{A}_q(\omega) = q\} = \{\varepsilon\}$, Then, the system $(\mathcal{Q}, \mathcal{U}, \Omega, \mathcal{A})$ is externally nonholonomic at $q \in \mathcal{Q}$ if and only if $a(\widehat{\Omega}_q) \subsetneqq a(\widetilde{\Omega}_q)$, where the inclusion is a group inclusion.*

Proof. By the hypothesis we can identify $a(\widetilde{\Omega}_q)$ with a subgroup of $([q], \cdot)$ and write $\mathcal{R}_q(\widetilde{\Omega}_q) = a(\widetilde{\Omega}_q) \cdot q$. Then $a(\widetilde{\Omega}_q) \supsetneqq a(\widehat{\Omega}_q)$ if and only if $\mathcal{R}_q(\widetilde{\Omega}_q) \supsetneqq \mathcal{R}_q(\widehat{\Omega}_q)$.

4.2 Internal Nonholonomy

We restrict to driftless invertible systems where the inverse is defined uniquely, which is tantamount to assuming that $\widetilde{\Omega}_q$ is a group. Assume also that $\widetilde{\Omega}_q$ is finitely generated and denote by $S = \{s_1, \ldots, s_n\}$ a set of generators.

Consider now the subset Ω_q^S of *simple* input words over S, i.e. those strings that either include a generator, or its inverse, but not both. More precisely, let

$$\widetilde{\Omega}_q^S = \{s_{\sigma(1)}^{k_{\sigma(1)}} s_{\sigma(2)}^{k_{\sigma(2)}} \ldots s_{\sigma(n)}^{k_{\sigma(n)}} : \sigma \in \mathcal{P}(n), k_{\sigma(j)} \in \mathbb{Z},\ j = 1, \ldots, n\}$$

where $k_{\sigma(i)}$ is the number of times the symbol $s_{\sigma(i)}$ is used (negative values meaning that $\bar{s}_{\sigma(i)}$ is used instead), and $\mathcal{P}(n)$ is the set of permutations of $(1, 2, \ldots, n)$. Let $\mathcal{R}_q(\widetilde{\Omega}_q)$ and $\mathcal{R}_q(\widetilde{\Omega}_q^S)$ denote the reachable set from q under input streams in $\widetilde{\Omega}_q$ and in $\widetilde{\Omega}_q^S$, respectively. Definitions we propose to capture the second type of nonholonomy are then as follows:

Definition 2. *A system $(\mathcal{Q}, \mathcal{U}, \Omega, \mathcal{A})$ is said to be noncommutative at $q \in \mathcal{Q}$ if $\widetilde{\Omega}_q$ contains at least two elements ω_1 and ω_2 such that for their* commutator *$[\omega_1, \omega_2] := \omega_1 \omega_2 \bar{\omega}_1 \bar{\omega}_2$ it holds $\mathcal{A}_q([\omega_1, \omega_2]) \neq q$.*
A system is internally nonholonomic *at q if there exists a set of generators S and $\omega_1, \omega_2 \in \widetilde{\Omega}_q^S$ such that $\mathcal{A}_q([\omega_1, \omega_2]) \notin \mathcal{R}_q(\widetilde{\Omega}_q^S)$.*

Clearly, this definition tends to generalize upon the second observation made in the introduction about classic nonholonomic systems, i.e. noncommutativity of vector fields.

The two notions of nonholonomy have a suggestive geometric interpretation (see fig.3), which is reminiscent of Berry's phase in quantum mechanics [25]. Berry noticed that if a quantum system evolves in a closed path in its parameter space, after one period the system would return to its initial state, however with a multiplicative phase containing a term depending only upon the geometry of the path the system traced out, or Berry's Phase. In our setting, consider a local decomposition of $\mathcal{Q}$ in a *base* space $\mathcal{B}$ and a *fiber* space $\mathcal{F}$, with $\mathcal{B} \times \mathcal{F} = \mathcal{Q}$. Choosing coordinates $q = (q_B, q_F)$ and denoting the canonical projections $\Pi_{\mathcal{B}}(q) = q_B$, $\Pi_{\mathcal{F}}(q) = q_F$, let $\mathcal{B}$ be a maximal codimension set such that $\Pi_F(\mathcal{R}_q(\widetilde{\Omega}_q^{[q]})$ (for external nonholonomy), or $\Pi_F(\mathcal{R}_q(\widetilde{\Omega}_q^S)$ (for internal nonholonomy), are constant. If there exists an input stream which would steer the system from q to $q^\star$ with $\Pi_B(q) = \Pi_B(q^\star)$ but $q \neq q^\star$, then the system is nonholonomic at q, and the difference between $\Pi_F(q^\star)$ and $\Pi_F(q)$ is the corresponding holonomy phase.

As regards tests for checking internal nonholonomy, we notice explicitly that an equivalent statement of internal nonholonomy is $\mathcal{R}_q(\widetilde{\Omega}_q^S) \subsetneq \mathcal{R}_q(\widetilde{\Omega}_q)$. The situation is quite different from external nonholonomy, though, because the comparison among such reachable sets can not be lifted to a comparison among groups. Indeed, $\widetilde{\Omega}_q^S$ lacks a group structure (the composition of simple words is not simple in general), and, by definition, the group generated by $\widetilde{\Omega}_q^S$ is the whole $\widetilde{\Omega}_q$. In the following, we provide a characterization of internal nonholonomy for systems of type (3).

We should like first to compare the traditional notions of nonholonomy with our more general definitions, showing that the former are particular cases of the

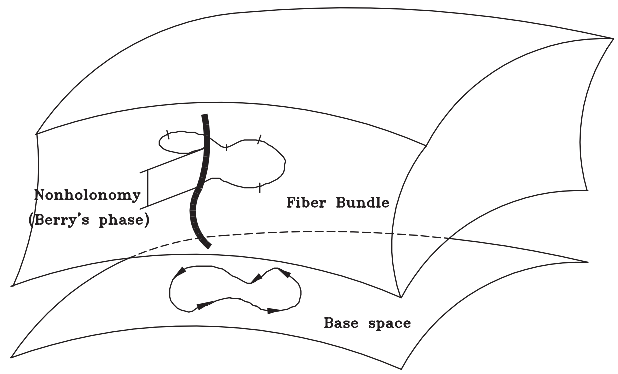

Fig. 3. Illustrating the definition of nonholonomic systems

latter. In particular, recall that the smooth, continuous time system

$$\dot{q} = G(q)u, \quad q \in \mathcal{Q} = \mathbb{R}^n, \ \mathbf{u} \in \mathcal{U} = \mathbb{R}^m \tag{3}$$

is nonholonomic in the classic sense iff $\dim(Lie_q) > \dim(\Delta_q)$, where $\Delta_q := \mathsf{span}\,\{G(q)u : u \in \mathcal{U}\}$ denotes the distribution generated by $G(q,u)$, while Lie_q is the Lie Algebra of system (3) evaluated at q.

Remark 1. Notice that the classic concept of nonholonomy is intrinsically local, being indeed equivalent to a notion of small-time local (or local-local) controllability. Such local character of classical nonholonomy is not reflected in the above definition of internal nonholonomy, as shown by the example below.

Example 5 Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} v\, w \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ \alpha + \varphi(q - \bar{q}) \end{bmatrix} u_2$$

with $(u_1, u_2) \in \mathbb{R}^2$ and $(x_1, x_2) \in T^2$, the two dimensional torus identified with $\mathbb{R}^2$ quotient the equivalence relation $(x_1, x_2) \sim (x_1 + k_1, x_2 + k_2)$, $k_1, k_2 \in \mathbb{Z}$. Fix $\bar{q} \in T^2$, take α a constant $\alpha \notin \mathbb{Q}$, and $\varphi(\cdot)$ a function with $\varphi(0) = 0$ and $\nabla\varphi(0) = (0, 1)$. Observe that $[v, w](\bar{q}) = (0, \alpha)$, thus $\dim(\Delta_{\bar{q}}) = 1$, $\dim(Lie_{\bar{q}}) = 2$, so the system is classically nonholonomic at $\bar{q}$.
However the system is not internally nonholonomic. Indeed, taking two generators that make the system flow along the vector fields v and w respectively, the set

$$\mathcal{R}_{\bar{q}}(\widetilde{\Omega}_{\bar{q}}^S) = \left\{ e^{tv} e^{sw} \bar{q} : \, t, s \in \mathbb{R} \right\}$$

coincide with T^2.

With such a motivation, consider a neighborhood $R \subseteq [q]$ of q, and the set $\mathcal{R}_q^R(\widetilde{\Omega}_q^S)$ of configurations reachable from q under input streams in $\widetilde{\Omega}_q^S$, restricted so that the trajectory from q does not leave R. A system will be said internally nonholonomic at q *with respect to* R, if $\mathcal{A}_q([\omega_1, \omega_2]) \notin \mathcal{R}_q^R(\widetilde{\Omega}_q^S)$. When this holds for arbitrarily small R, the system is said to be *locally* internally nonholonomic.

Proposition 3. *A smooth, continuous time system (3) which is nonholonomic in the classic sense, is locally internally nonholonomic.*

Proof. Clearly, there is only one equivalence class $[q] = \mathcal{Q}$ in this case, and $\widetilde{\Omega}$ contains all input functions. We assume, without loss of generality (cf. [26]), that the columns of G are independent and that $\widetilde{\Omega}$ is comprised of actions corresponding to piecewise constant functions $\mathbb{R}^+ \mapsto \mathbb{R}^m$.

A set of generators for $\widetilde{\Omega}$ can be written as $S = (s_1, \ldots, s_m)$, $s_i = w_i e_i$, where e_i denotes the i–th column of the $m \times m$ identity matrix, and w_i is the (δ, τ) window function,

$$w_i(t) = \begin{cases} 0, & t < 0 \\ \delta_i, & 0 \leq t < \tau_i \\ 0, & \tau_i \leq t. \end{cases}$$

Notice that both the amplitude $\delta_i \in \mathbb{R}$ and duration $\tau_i \in \mathbb{R}^+$ of the window functions in the generators are considered free, hence each generator is a two-parameter family of finite-support, constant functions. The corresponding actions are given by

$$\begin{aligned} a(\widetilde{\Omega}_q^S) = \{ & e^{F_{\sigma(1)}} e^{F_{\sigma(2)}} \ldots e^{F_{\sigma(n)}}, \sigma \in \mathcal{P}(n), F_i = \tau_i \delta_i G e_i, \\ & \delta_i \in \mathbb{R}, \tau_i \in \mathbb{R}^+, \ i = 1, \ldots, n \}, \end{aligned}$$

where $e^{F_j} = e^{\tau_j \delta_j G e_j}$ is the formal exponential flow along the j-th control vector field. Therefore, the tangent space to the set of points locally reachable from q by simple words is given by Δ_q. Since there exists a bracket $[Ge_i, Ge_j](q)$ not contained in Δ_q, then the action of the commutator $(s_i s_j \bar{s}_i \bar{s}_j)$ generating this bracket proves the desired result.

Remark 2. The definition of internal nonholonomy requires the existence of a set of generators for which the action of commutators can not be obtained by simple actions. In general, not all sets of generators are suitable to show internal nonholonomy for a system, as is in general required that the generator set satisfies a minimality condition (this is illustrated in the example 1-iii) in the next section). We therefore introduce the following

Definition 3. *A system of generators S is* minimal *if $\#(S) = \dim(\Delta_q)$, where $\#(S)$ is the cardinality of S.*

Define a matrix $W = \left[W_1 | \cdots | W_p \right] \in \mathbb{R}^{m \times p}$, with $rank(W) = m$, and consider a quantized input set

$$\mathcal{U} = \{0, \pm W_1, \ldots, \pm W_p\} \subset \mathbb{R}^m. \tag{4}$$

Proposition 4. *The system*

$$\dot{q} = G(q)u, \quad q \in \mathcal{Q} = \mathbb{R}^n, \ u \in \mathcal{U} \subset \mathbb{R}^m \tag{5}$$

with $\dim(Lie_q) > \dim(\Delta_q)$, and $\mathcal{U}$ a final set as in (4) is locally internally nonholonomic.

Proof. Again $[q] = \mathcal{Q}$ here, and $\widetilde{\Omega} = \mathcal{U}^*$. Let $S = (s_1, \ldots, s_m)$, $s_j = W_{i_j}$, $j = 1, \ldots, m$, $i_j \in \{1, \ldots, p\}$, be a minimal set of generators. Recall the Baker-Campbell-Hausdorff formula:

$$e^{t_n GW_{\sigma(n)}} \cdots e^{t_1 GW_{\sigma(1)}} =$$
$$\exp\left(\sum_{i=1}^n t_i GW_{\sigma(i)} + \frac{1}{2}\sum_{i<j} t_i t_j [GW_{\sigma(i)}, GW_{\sigma(j)}] + o(t^4)\right).$$

Since the set of generators is minimal, we have that $G(q)W_i$ are independent vectors. Hence the term $\sum_i t_i G(q) W_i$ vanishes only for all t_i equal to zero. We obtain that simple word actions can locally reach points only along the directions of $\Delta(q)$. Since the system is classically nonholonomic there exists a bracket $[GW_i, GW_j](q)$ not contained in Δ_q, hence the commutator action generating this bracket gives the desired result.

We now give some definition and results to provide a partial converse of Propositions 3, 4. Consider a system described by the dynamics in (5), with quantized input set.

We need to introduce the following

Definition 4. *A system of type (3),(5) is* strongly *internally nonholonomic if $T_q\mathcal{R}_q^R(\widetilde{\Omega}) \supsetneqq T_q\mathcal{R}_q^R(\widetilde{\Omega}^S)$, where by $T_q\mathcal{R}_q^R(\widetilde{\Omega})$ (resp. $T_q\mathcal{R}_q^R(\widetilde{\Omega}^S)$) we denote the tangent space at q to the set of reachable points from q using controls in $\widetilde{\Omega}$ (resp. $\widetilde{\Omega}_q^S$) so that the trajectory from q does not leave a small neighborhood R of q.*

We immediately have

Proposition 5. *If a system of type (3),(5) is strongly internally nonholonomic, then it is internally nonholonomic.*

Proposition 6. *A strongly internally nonholonomic system of type (3) or (5) is classically nonholonomic.*

Proof. The above proposition follows directly from the definition. Indeed, by absurd, consider a holonomic system, then the flow along any bracket of any order is given by the flow along some direction in the vector space generated by the columns of $G(q)$. Then $T_q\mathcal{R}_q^R(\widetilde{\Omega}) = T_q\mathcal{R}_q^R(\widetilde{\Omega}^S)$.

5 Examples revisited

Example 1 - i). Internal nonholonomy of this system follows directly shown by proposition 3. It is interesting however to report a direct constructive proof in this case, obtained by taking the input construction commonly used in textbooks to illustrate "lie-bracket motions" (see e.g. [5]). Namely, let $S = (s_1, s_2)$ with $s_1(t) = (\delta_1\ 0)$, $t \in [0, \tau_1)$ and $s_2(t) = (0\ \delta_2)$, $t \in [0, \tau_2)$ (hence $\bar{s}_i = -s_i$, $i = 1, 2$). One easily gets $\mathcal{R}_{q_0}(\widetilde{\Omega}^S) = (x_0 + \alpha,\ y_0 + \beta,\ z_0 - y_0\alpha + x_0\beta + \alpha\beta), \alpha, \beta \in \mathbb{R}$, while $\mathcal{A}_{q_0}(s_1 s_2 \bar{s}_1 \bar{s}_2) = (x_0, y_0, z_0 + 2\delta_1\delta_2\tau_1\tau_2)$. Hence $\mathcal{A}_{q_0}([s_1, s_2]) \notin \mathcal{R}_{q_0}(\widetilde{\Omega}^S)$.

Example 1 - ii). Definition (2) equally applies in discrete time. This can be shown by taking e.g. $s_1 = (\delta_1\ 0)$, $s_2 = (0\ \delta_2)$, so that $\mathcal{A}_{q_0}([s_1, s_2]) =$

$(x_0, y_0, z_0 + 2\delta_1\delta_2)$, while $\mathcal{R}_{q_0}$ is as before. The continuity of the control set guarantees complete reachability for this system in both the continuous and discrete time cases.

Example 1 - iii). The restriction on controls does not substantially change the analysis under continuous time. Indeed, considering $s_1(t) = (a\ 0)$, $t \in [t_1, t_1+\tau_1]$, $s_2(t) = (0\ b)$, $t \in [t_2, t_2 + \tau_2]$, one gets $\mathcal{A}_{q_0}([s_1, s_2]) = (x_0, y_0, z_0 + 2ab\tau_1\tau_2)$, and both nonholonomy and complete reachability easily follow from arbitrarity of τ_1, τ_2.

We want to underscore here how a non-minimal choice of generators would not lead to a conclusion. Take for instance

$$\mathcal{U} = \{0, \pm(1,0), \pm(-1/2, 1/2),\ \pm(-1/2, -1/2)\}\,.$$

The actions corresponding to the above controls give the flows along the vector fields: $v_1 = (1, 0, -y)$, $v_2 = \frac{1}{2}(-1, 1, x + y)$, $v_3 = \frac{1}{2}(-1, -1, y - x)$. Moreover the action corresponding to a simple input stream is given by

$$e^{t_3 v_{\sigma(3)}} e^{t_2 v_{\sigma(2)}} e^{t_1 v_{\sigma(1)}} = \exp\left(\sum_{i=1}^{3} t_i v_{\sigma(i)} + \frac{1}{2}\sum_{i<j} t_i t_j [v_{\sigma(i)}, v_{\sigma(j)}] + o(t^4)\right)$$

where the above equivalence is given by the Campbell-Baker-Hausdorff formula. Computing the Lie brackets $[v_i, v_j]$, $i, j = 1, 2, 3$ gives

$$[v_1, v_2] = (0, 0, 1),\ [v_1, v_3] = (0, 0, -1),\ [v_2, v_3] = (0, 0, 1),$$

hence the system is classically nonholonomic. However, this set of generators does not show internal nonholonomy, since it is possible to reach any point (p_1, p_2, p_3) in a neighborhood of the origin by the action of a simple input stream. Indeed, up to higher order terms, it is enough to solve one of the two systems

$$\begin{cases} (p_1, p_2, 0) = t_1 v_1 + t_2 v_2 + t_3 v_3 \\ (0, 0, p_3) = t_1 t_2 [v_1, v_2] + t_1 t_3 [v_1, v_3] + t_2 t_3 [v_2, v_3] \end{cases}$$

or

$$\begin{cases} (p_1, p_2, 0) = t_1 v_1 + t_2 v_3 + t_3 v_2 \\ (0, 0, p_3) = t_1 t_2 [v_1, v_3] + t_1 t_3 [v_1, v_2] + t_2 t_3 [v_3, v_2] \end{cases}$$

obtained via simple stream actions for the choice $\sigma = (1)$ and $\sigma = (23)$ respectively. From the first system we get

$$\begin{array}{l} t_1 = t_3 + p_1 + p_2 \\ t_2 = t_3 + 2p_2 \\ t_3^2 + 4p_2 t_3 - \gamma = 0 \\ \gamma = 2p_3 - 2p_2(p_1 + p_2) \end{array}$$

while from the second we obtain:

$$\begin{array}{l} t_1 = t_3 + p_1 - p_2 \\ t_2 = t_3 - 2p_2 \\ t_3^2 - 4p_2 t_3 + \gamma = 0 \\ \gamma = 2p_3 - 2p_2(p_1 + p_2). \end{array}$$

Hence at least one of the two systems has a real solution.

Example 1 - iv). In the discrete input, discrete time case, the input commutator $[s_1, s_2]$ with $s_1 = (a,\ 0)$, $s_2 = (0\ b)$, produces $\mathcal{A}_{q_0}([s_1, s_2]) = (x_0, y_0, z_0+2ab)$. Internal nonholonomy is maintained. However, the reachable set from the origin is only comprised of configurations in a discrete set, $\mathcal{R}_0 = \{q\ :\ x = \ell a,\, y = mb,\, z = nab,\ \ \ell, m, n \in \mathbb{Z}\}$. The situation is completely different, and density of the reachable set is guaranteed, if e.g. $\mathcal{U} = \{(u_1, u_2)|\ u_1 \in \{0, a, -a, c, -c\},\ u_2 \in \{0, b, -b, d, -d\},\ \ a, b, c, d \in \mathbb{R}\}$ with $\frac{a}{c}, \frac{b}{d} \not\in \mathbb{Q}$.

The interpretation of nonholonomy given in fig.3 applies to all cases above, using coordinates x, y to describe the base space, while z parameterizes the fiber.

Example 2. Two input equivalence classes are defined in $\mathcal{Q}$ as $[q]_{free} = \{q \in \mathcal{Q} : y < y_0\}$ and $[q]_{engaged} = \{q \in \mathcal{Q} : y \geq y_0\}$. Clearly, $\mathcal{R}_{q_0}(\widehat{\Omega}_{q_0}) = \{(x, y, z) \in \mathcal{Q} : z = z_0\}$, for all $q_0 = (x_0, y_0, z_0) \in [q]_{free}$, while $\mathcal{R}_{q_0}(\widetilde{\Omega}_{q_0}) = \mathbb{R}^3$. The system is thus externally nonholonomic according to definition (1).

Interestingly enough, the system is not internally nonholonomic as per definition (2). Indeed, to generate the set $\widetilde{\Omega}_{q_0}$, at least two types of streams must be considered: an internal type e.g. $s_i : (x_0, y_0, z_0) \mapsto (x, y, z_0)$, and an external type (taking the state out of $[q]_{free}$ temporarily), e.g. $s_e : (x_0, y_0, z_0) \mapsto (x', y', z')$. Clearly, simple streams over this set of generators are sufficient to reach any configuration of the system ($\mathcal{R}_q(\widetilde{\Omega}^S_q) = \mathbb{R}^3$), hence internal nonholonomy does not apply.

Base variables for this example would be x and y, while z represents the fiber variable. Rectification of motion is obtained by holonomic phase accumulation in succesive cycles. By changing frequency and phase of the inputs, different directions and velocities of the rod motion can be achieved. Note in particular that input u_2 need not actually to be finely tuned, as long as it is periodic, and it could be chosen as a resonant mode of the vibrating actuator: tuning only u_2 still guarantees in this case the (non-local) reachability of the system (cf. [27, 11]).

Example 3: In the rolling polyhedron system let a configuration be described by the index, position and orientation of the face currently on the plane, as $q = (F, x, \theta) \in \mathcal{F} \times \mathbb{R}^2 \times S^1$, with $\mathcal{F} = \{F_1, \ldots, F_n\}$ the set of faces. For each face is associated to a finite set of adjacent faces, there are n input equivalence classes given by $[q] = \{F = F_i, x \in \mathbb{R}^2, \theta \in S^1\}$. This system verifies the hypothesis of proposition 2, because we can identify a class $[q]$ with a subgroup of isometries of the plane $\mathbb{R}^2 \oplus S^1$. External nonholonomy can be proved quite straightforwardly in this case by comparing the action of the group $\mathcal{R}_q(\widetilde{\Omega}_q)$ of sequences of faces starting and ending with F_i, with $\mathcal{R}_q(\widehat{\Omega}_q) = \emptyset$.

Internal nonholonomy according to definition 2 also holds: indeed, $\widetilde{\Omega}_q$, the set of words that bring back the polyhedron on the same face lying on the plane, is generated (see [28]) by the finite set $S = \{R_\lambda,\ \lambda = 1, \ldots, h-1\}$, where R_λ is a planar rotation of an angle equal to the polyhedron defect angle β_λ at the λ–th vertex, centered at the point corresponding with that vertex in the planar development of the polyhedron. If $\beta_\lambda/\pi \in \mathbb{Q}$ for all $\lambda = 1, \ldots, h-1$, then $\widetilde{\Omega}^S_q$ is a finite set because, if $\beta_\lambda = 2\pi \frac{m_\lambda}{p_\lambda}$, $R_\lambda^{p_\lambda} = (0, 0)$. Therefore $\mathcal{R}_q(\widetilde{\Omega}^S_q)$ is a finite set. Since $\mathcal{R}_q(\widetilde{\Omega}_q)$ is an infinite countable set, nonholonomy immediately follows. If, otherwise, there exists λ such that $\beta_\lambda/\pi \notin \mathbb{Q}$ then there exists another index λ', $\lambda' \neq \lambda$ for which it also holds $\beta_{\lambda'}/\pi \notin \mathbb{Q}$. Without loss of generality we can assume $\lambda = 1$ and $\lambda' = 2$ and choose the set of $h-1$ generators given by $\beta_2, ...\beta_h$. In order to prove nonholonmy we have to compare commutators with translations in $\widetilde{\Omega}_q^S$. Translations in $\widetilde{\Omega}_q^S$ are written as $R_{\sigma(2)}^{k_{\sigma(2)}} R_{\sigma(3)}^{k_{\sigma(3)}} \dots R_{\sigma(h)}^{k_{\sigma(h)}}$, with $k_{\sigma(j)} = 0$ if $\beta_{\sigma(j)}/\pi \notin \mathbb{Q}$. In other words translations in $\widetilde{\Omega}_q^S$ have to be generated only by those generators with λ such that β_λ is irrational with π. Now, let t be any translation in $\widetilde{\Omega}_q^S$. Then the commutator $[R_2, t]$ gives a translation of $t(e^{-\jmath\beta_2} - 1)$ which cannot be generated by simple words.

6 Conclusions

The notions of nonholonomy and reachability are conventionally related to differentiable control systems, and are defined in terms of their differential geometric properties. However, these notions apply also to more general systems, including discrete and hybrid systems. In this paper, we have given a generalization of the concept of nonholonomy to such classes of systems. By studying the intimate nature of a few carefully chosen examples, we observed two different aspects of the hybrid nonholonomic phenomenon, which have been captured in the notions of internal and external nonholonomy. We also provided some tools for investigating the applicability of definitions to given systems. However, much work remains to be done in that direction.

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