Course on Model Predictive Control
Part I – Introduction

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Outline

1. Introduction to MPC: motivations and history
2. Comparison with conventional feedback control
3. Simple example and typical industrial architecture
4. Some reminders of linear systems theory and optimal control/estimation

Brief history of Model Predictive Control

Origins and motivations

- Model Predictive Control (MPC) algorithms were born in industrial environments (mostly refining companies) during the 70's:
  - DMC (Shell, USA) [Cutler and Ramaker, 1979]
  - IDCOM (Adersa-Gerbios, France) [Richalet et al., 1978]
- Necessity to satisfy the more **stringent** production requests, e.g.:
  - economic **optimization**
  - **maximum exploitation** of production capacities
  - **minimum variability** in product qualities

Brief history of Model Predictive Control (cont'd)

Industry and academia

- Nowadays, most **complex plants** especially in refining and (petro)chemical industries use MPC systems
- After an initial **reluctance**, the academia "embraced" MPC contributing to:
  - establish theoretical foundations
  - develop new algorithms

Commercial product evolution

Commercial Products (partial list updated to 1996)

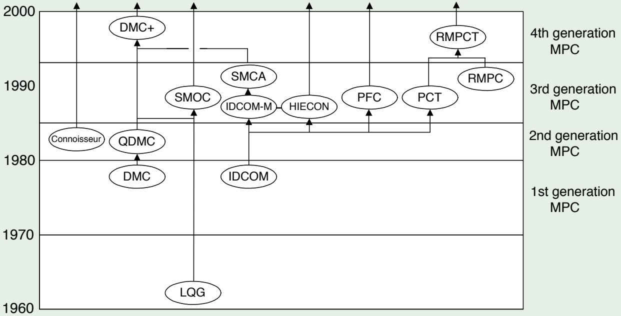
- [DMC] Dynamic Matrix Control ⇒ DMC Corporation (USA)
- [SMCA] (ex IDCOM) Multivariable Control Architecture ⇒ Set-Point,Inc. (USA)
- [PCT] (RMPCT) Predictive Control Technology ⇒ Honeywell - Profimatics (USA)
- [OPC] Optimum Predictive Control ⇒ Treiber Controls, Inc. (Canada)
- [MVPC] Multivariable Predictive Control ⇒ ABB Ind. System Corp. (USA)
- [IDCOM-Y] ⇒ Johnson Yokogawa Corp. (USA)
- [MVC] Multivariable Control ⇒ Continental Control, Inc. (USA)
- [C-MCC] Contas-Multivariable Constrained Control ⇒ CONTAS s.r.l. (Italy)

Commercial product evolution

Merges

- In middle of the 90's, many **acquisitions** and **merges** occurred
- The situation became quite steady with **two main competitors** (DMC+ and RMPCT) and other less diffused technologies (Connoisseur, SMOC, PFC, etc.)

From [Qin and Badgwell, 2003]



Keywords

MPC keywords

In most commercial product acronyms we find several important keywords that define the MPC technologies

- Control
- Model
- Predictive
- Multivariable
- Robustness
- Constraints
- Optimization
- Identification

Analysis of such characteristic features in **comparison** with **conventional control** schemes

Conventional feedback control (PID)

Essential features

- Control action based on the **tracking error**, $e(t) = y_{sp}(t) - y(t)$ (no prediction)
- **Fixed structure** regulator (e.g., PID)

$$u(t) = K_c e(t) + \frac{K_c}{\tau_I}\int_0^t e(\tau)d\tau + K_c \tau_D \frac{de}{dt}$$

- **Constraints:** only on the manipulated variable (absolute or incremental)

$$u_{\min} \leq u(t) \leq u_{\max}, \qquad \left|\frac{du}{dt}\right| \leq \Delta u_{\max}$$

- **Process model:** "sometimes" used to define the tuning parameters K_c, τ_I, τ_D
- **Optimization**: no direct optimization is achieved (only by tuning)

Shortcomings of conventional feedback control (PID)

Issues

Conventional feedback controllers are **not able** to face:

- **Interactions** from each manipulated variable to all controlled variables
- **Directionality**

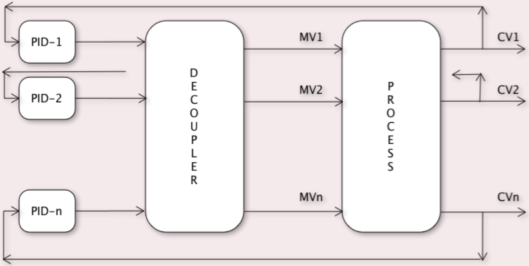
  *Certain **combinations** of control actions have a much larger (20-200 times) effect on the controlled variables than other combinations of the same control actions. Thus:*

  Perturbations in the former directions are rejected much more easily than perturbations in the latter directions.

- **Constraints** on the controlled variables (e.g., product qualities)
- **Optimization** of the overall plant (nonsquare systems)

Conventional multivariable control

Typical structure



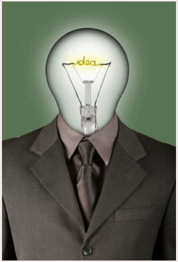
Features and limitations

- The **decoupler structure** (model based) determines the achievable performances (interactions and directionality)
- Decoupler **robustness** is an issue
- When # CV $\neq$ # MV (**nonsquare systems**) different **alternatives** are necessary (split-range, selective control, etc.)

Main features of MPC

MPC became a *successful technology* due to the following features:

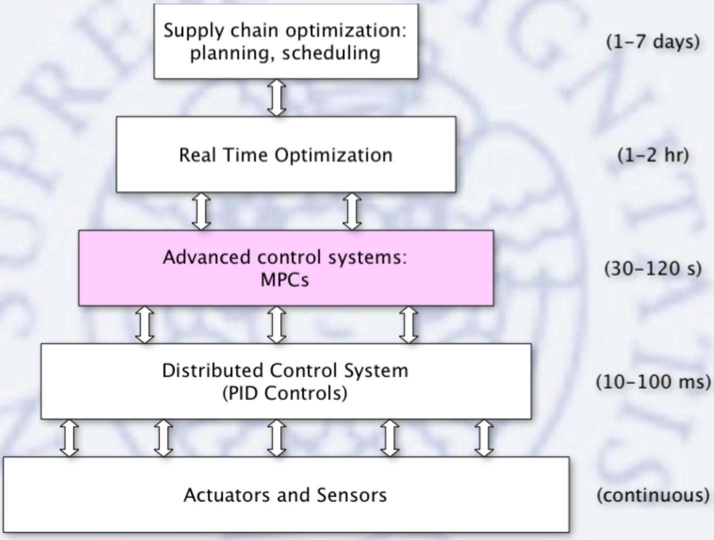
- ease of handling **multivariable systems**
- ease of handling **complicated dynamics** (e.g., delays, inverse response, ramps, etc.)
- ease of handling **constraints** on controlled and manipulated variables (pushing the **plant towards its limits**)
- straightforward applicability to **feedforward information** (measurable disturbances)



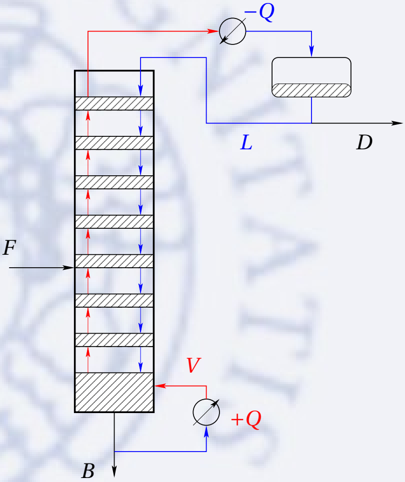
Industrial applications [Qin and Badgwell, 2003]

Area	Aspen Technology	Honeywell Hi-Spec	Adersa	PCL	MDC	Total
Refining	1200	480	280	25		1985
Petrochemicals	450	80	–	20		550
Chemicals	100	20	3	21		144
Pulp and Paper	18	50	–	–		68
Air & Gas	–	10	–	–		10
Utility	–	10	–	4		14
Mining/Metallurgy	8	6	7	6		37
Food Processing	–	–	41	10		51
Polymer	17	–	–	–		17
Furnaces	–	–	42	3		45
Aerospace/Defense	–	–	13	–		13
Automotive	–	–	7	–		7
Unclassified	40	40	1045	26	450	1601
Total	1833	696	1438	125	450	4542
First App.	DMC:1985 IDCOM-M:1987 OPC:1987	PCT:1984 RMPCT:1991	IDCOM:1973 HIECON:1986	PCL: 1984	SMOC: 1988	
Largest App	603×283	225×85		–	31×12	–

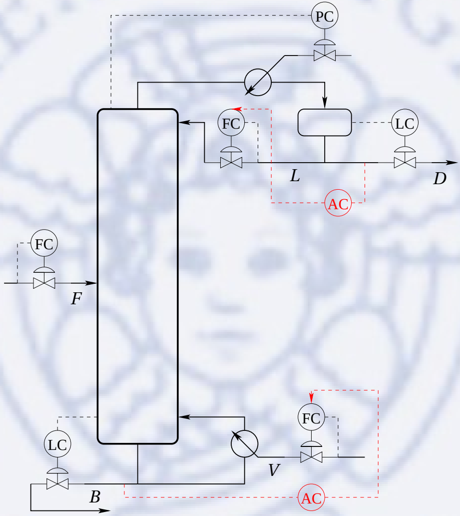
Hierarchy of an optimization and control system



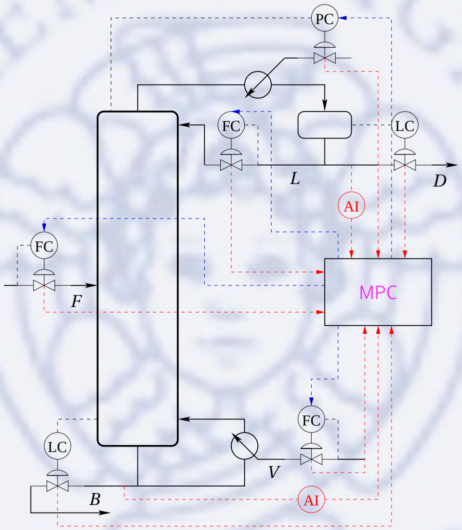
Typical example: a distillation unit



Conventional decentralized control of a distillation unit

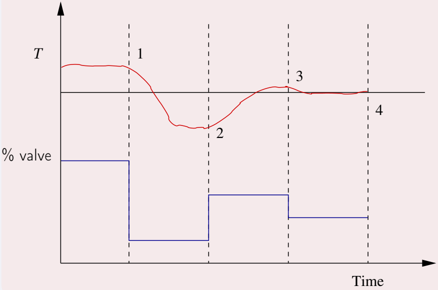


MPC of a distillation unit



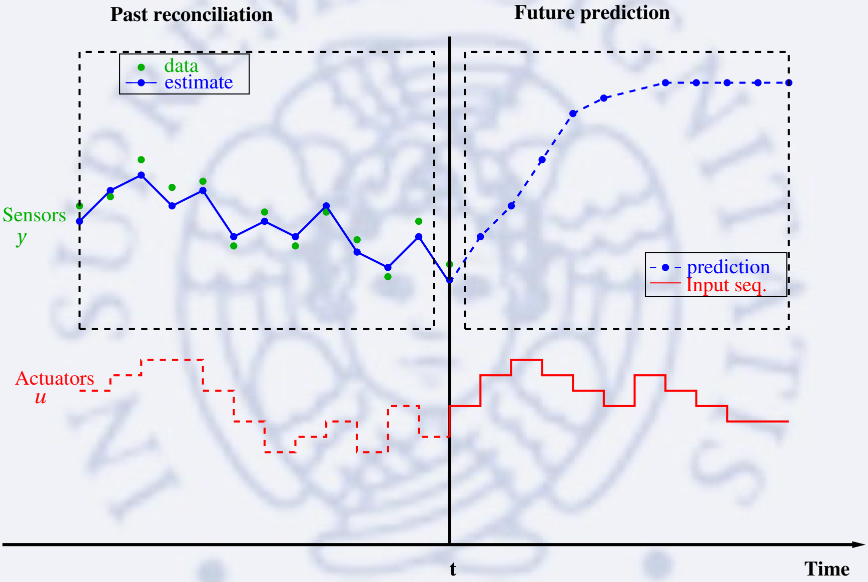
MPC: basic idea

Manual control of a furnace temperature

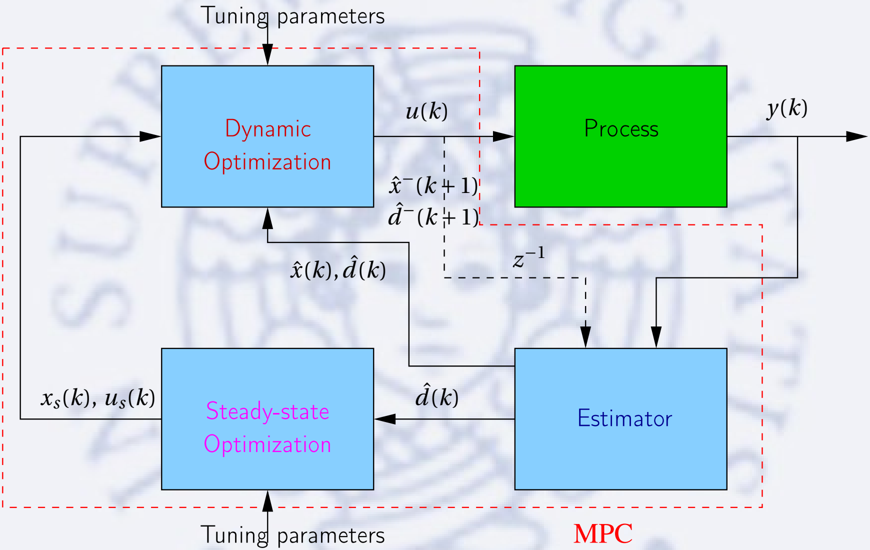


- Use the **process model** (DC gain)
- Feedback information is the difference between **actual** and **predicted** process output
- Actions are **iterated** based on feedback information

MPC framework



General structure of an MPC algorithm



Linear dynamic models: continuous-time

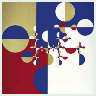
State-space formulation (LTV)

$$\frac{dx}{dt} = A(t)x(t) + B(t)u(t) \qquad x \in \mathbb{R}^n$$
$$y(t) = C(t)x(t) + D(t)u(t) \qquad u \in \mathbb{R}^m$$
$$x(0) = x_0 \qquad y \in \mathbb{R}^p$$

In **most** applications $D(t) = 0$

State-space formulation (LTI)

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$



Solution:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Linear dynamic models: discrete-time

State-space formulation (LTV)

$$
\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)u(k) \qquad & x \in \mathbb{R}^n \\
y(k) &= C(k)x(k) + D(k)u(k) \qquad & u \in \mathbb{R}^m \\
x(0) &= x_0 \qquad & y \in \mathbb{R}^p
\end{aligned}
$$

State-space formulation (LTI)

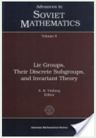
$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
$$

or simply:

$$
\begin{aligned}
x^+ &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
$$

Solution:

$$
x(k) = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)
$$



Linear Quadratic Regulation problem

Problem setup

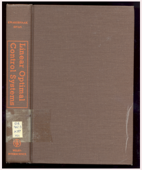
- Discrete-time LTI system $x^+ = Ax + Bu$
- Consider N time steps into the future, collect input sequence

$$\mathbf{u} = \{u(0),\ u(1),\ ,\ldots,\ u(N-1)\}$$

- Define the cost function:

$$V_N(x(0), \mathbf{u}) = \frac{1}{2} \sum_{k=0}^{N-1} \left[x(k)' Q x(k) + u(k)' R u(k) \right] + \frac{1}{2} x(N)' P_f x(N)$$

$$\text{subject to: } x^+ = Ax + Bu$$



Optimal LQ control problem

$$\min_{\mathbf{u}} V_N(x(0), \mathbf{u})$$

Optimizing multi-stage functions

Basic idea

Solve the following **problem** of **three variables** (x, y, z):

$$\min_{x,y,z} f(w,x) + g(x,y) + h(y,z), \qquad w \text{ fixed}$$

Rewrite as **three single-variable** problems:

$$\min_{x} \left[f(w,x) + \min_{y} \left[g(x,y) + \min_{z} h(y,z) \right] \right]$$

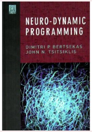
Iterative strategy

- Solve the **most inner** problem **first**: $\underline{h}^0(y) = \min_z h(y,z)$
- Proceed to the **intermediate** problem: $\underline{g}^0(x) = \min_y g(x,y) + \underline{h}^0(y)$
- Solve the **most outer** problem: $\underline{f}^0(w) = \min_x f(x,y) + \underline{g}^0(x)$

Principle of dynamic programming applied to LQR problem

- Let $\ell(x,u) = 1/2(x'Qx + u'Ru)$ and $\ell_N(x) = 1/2x'P_f x$
- Optimize over $u(N-1)$ **and** $x(N)$

$$\min_{u(0),x(1),\ldots,u(N-2),x(N-1)} \sum_{k=0}^{N-2} \ell(x(k),u(k)) + \underbrace{\min_{u(N-1),x(N)} \ell(x(N-1),u(N-1)) + \ell_N(x(N))}_{\text{Solve this first s.t. } x(N) = Ax(N-1)+Bu(N-1)}$$



- **Obtain**:
  $u^0(N-1) = K_N(N-1)x(N-1)$, with $K_N(N-1) = -(B'P_f B+R)^{-1}B'P_f A$
- Repeat to obtain the (backward) **Riccati recursions**:

$$u^0(k) = K_N(k)x(k), \text{ with } K_N(k) = -(B'\Pi(k+1)B+R)^{-1}B'\Pi(k+1)A$$

$$\Pi(k-1) = Q + A'\Pi(k)A - A'\Pi(k)B(B'\Pi(k)B+R)^{-1}B'\Pi(k)A, \quad \Pi(N) = P_f$$

Infinite horizon LQR problem

A quote from [Kalman, 1960]

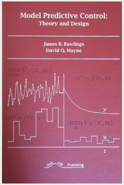
*In the engineering literature it is often assumed (**tacitly** and **incorrectly**) that a system with optimal control law is necessarily stable.*

Closed-loop with finite-horizon LQR

- Consider the optimal finite-horizon (N) control law: $u = K_N(0)x$
- Closed-loop system: $x^+ = Ax + Bu = (A + BK_N)x$
- Examples for which (see e.g. [Rawlings and Mayne, 2009]):

$$\max |\text{eig}(A + BK_N)| \geq 1$$

and hence the origin is **not asymptotically stable**



Infinite horizon LQR: let $N \to \infty$ and solve the Riccati equation

$$\Pi = Q + A'\Pi A - A'\Pi B(B'\Pi B + R)^{-1}B'\Pi A \Rightarrow K = -(B'\Pi B + R)^{-1}B'\Pi A$$

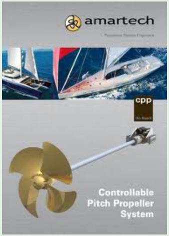
Controllability

Definition [Sontag, 1998]

A system $x^+ = Ax + Bu$ is **controllable** if for any pair of state y, z in $\mathbb{R}^n$, there exists a finite input sequence $\{u(0),\ u(1), \ldots, u(N-1)\}$ such that $x(0) = y$ implies $x(N) = z$

Tests for controllability

- Via **controllability matrix**: $\mathscr{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$
  (A, B) is controllable *iff* $\operatorname{rank}(\mathscr{C}) = n$
- Via **Hautus** Lemma – **conceptual**:
  $\operatorname{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \qquad \text{for all } \lambda \in \mathbb{C}$
- Via Hautus Lemma – **practical**:
  $\operatorname{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n \qquad \text{for all } \lambda \in \operatorname{eig}(A)$



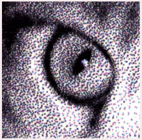
Infinite-horizon LQR and controllability

For (A, B) **controllable** and Q, R **positive definite**, there exists a **positive definite** solution of the Riccati equation, and the matrix $(A + BK)$ is **strictly Hurwitz**

Stochastic linear systems

Discrete-time LTI systems

$$x^+ = Ax + Gw$$
$$y = Cx + v$$
$$x(0) = x_0$$



$x(0)$, w and v are **random** variables

Gaussian assumption

We often make the following assumption:

$$x(0) \sim N(\bar{x}(0), P(0)), \quad w \sim N(0, Q), \quad v \sim N(0, R)$$

Notation: $x \sim N(\bar{x}, P)$ means that the random variable x is **normally distributed** with mean $\bar{x}$ and covariance P

Linear optimal state estimation

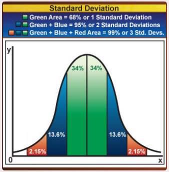
Preliminary results on normally distributed random variables

- If x and y are **n.d.** and (statistically) independent, i.e. $x \sim N(m_x, P_x)$ and $y \sim N(m_y, P_y)$, then the **joint density** is

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} P_x & 0 \\ 0 & P_y \end{bmatrix} \right)$$

- If x and y are **jointly n.d.**, i.e. $\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} P_x & P_{xy} \\ P'_{xy} & P_y \end{bmatrix} \right)$, then the **conditional density** of x given y, $(x|y)$, is:

$$(x|y) \sim N\left(m_x + P_{xy} P_y^{-1} (y - m_y), P_x - P_{xy} P_y^{-1} P'_{xy} \right)$$



- If $x \sim N(m_x, P)$ and $y = Cx$, then: $y \sim N(Cm_x, CPC')$
- If $x \sim N(m_x, P)$, $v \sim N(0, R)$ and $y = Cx + v$, then: $y \sim N(Cm_x, CPC' + R)$

Linear optimal state estimation (cont.'d)

Deriving the Kalman filter...

- Assume **prior knowledge**: $x(k) \sim N(\hat{x}^-(k), P^-(k))$
- Obtain **measurement** $y(k)$ that satisfies: $\begin{bmatrix} x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$
- Since $x(k)$ and $v(k)$ are **independent**, there holds:

$$\begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \sim N\left(\begin{bmatrix} \hat{x}^-(k) \\ C\hat{x}^-(k) \end{bmatrix}, \begin{bmatrix} P^-(k) & P^-(k)C' \\ CP^-(k) & CP^-(k)C'+R \end{bmatrix} \right)$$

- **Conditional density** $(x(k)|y(k)) \sim N(\hat{x}(k), P(k))$ with:
$\hat{x}(k) = \hat{x}^-(k) + L(k)\left(y(k) - C\hat{x}^-(k)\right)$
$L(k) = P^-(k)C'(CP^-(k)C' + R)^{-1}$
$P(k) = P^-(k) - P^-(k)C'(CP^-(k)C' + R)^{-1}P^-(k)C'$
- **Forecast** using $x(k+1) = Ax(k) + Gw(k)$

$$x(k+1) \sim N(\underbrace{A\hat{x}(k)}_{\hat{x}^-(k+1)}, \underbrace{AP(k)A' + GQG'}_{P^-(k+1)})$$

Linear optimal state estimation (cont.'d)

Convergence of the state estimator

Consider the **noise-free** system:

$$x(k+1) = Ax(k) + Bu(k), \qquad y = Cx(k)$$

Given an **incorrect initial estimate** $\hat{x}^{-}(0)$, we use a time-varying Kalman filter $L(k)$. **Is $\hat{x}^{-}(k) \to x(k)$ as $k \to \infty$?**

Estimation error and steady-state Kalman filter

- Define the **state estimation error**: $e(k) = x(k) - \hat{x}^{-}(k)$
- We obtain:

$$e(k+1) = (A - AL(k)C)\, e(k)$$

- Thus, $e(k) \to 0$ as $k \to \infty$ if $(A - ALC)$ is **strictly Hurwitz**, where:
  $L = \Pi C'(C\Pi C' + R)^{-1}$
  $\Pi = A\Pi A' - A\Pi C'(C\Pi C' + R)^{-1}\Pi C' A' + GQG'$

Definition [Sontag, 1998]

A system $x^+ = Ax + Bu$ with **measured output** $y = Cx$, is **observable** if there exists a finite N such that for any (**unknown**) initial state $x(0)$ and N **measurements** $\{y(0), y(1), \ldots, y(N-1)\}$, the initial state $x(0)$ can be **determined uniquely**

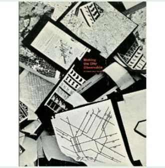
Tests for observability

- Via **observability matrix**: $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

  (A, C) is observable *iff* $\text{rank}(\mathcal{O}) = n$
- Via **Hautus** Lemma – **conceptual**:

  $$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \qquad \text{for all } \lambda \in \mathbb{C}$$
- Via Hautus Lemma – **practical**:

  $$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \qquad \text{for all } \lambda \in \text{eig}(A)$$



Regulator vs estimator

- Regulator:
  $x^+ = Ax + Bu, \quad y = Cx, \quad V_\infty(x(0), \mathbf{u}) = \frac{1}{2}\sum_{k=0}^{\infty}\left[x(k)'C'QCx(k) + u(k)'Ru(k)\right]$
- Estimator:
  $x^+ = Ax + Gw, \qquad y = Cx + v$

Duality

Regulator	Estimator
$R > 0$, $Q > 0$	$R > 0$, $Q > 0$
(A, B) controllable	(A, C) observable
(A, C) observable	(A, G) controllable
A	A'
B	C'
C	G'
Π	Π
K	$-(AL)'$
$A + BK$	$(A - ALC)'$
x	e'

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