Controllability for Pairs of Vehicles Maintaining Constant Distance

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***Abstract*—This paper studies the controllability of pairs of identical nonholonomic vehicles maintaining a constant distance. The study provides controllability results for the five most common types of robot vehicles: Dubins, Reeds-Shepp, differential drive, car-like and convexified Reeds-Shepp. The challenge of achieving controllability of such systems is that their admissible control domains depend on configuration variables. A theorem of controllability specifical for such systems has been obtained based on known controllability theorems. As a result, we show that pairs of the latter three types are completely controllable, i.e. can be steered between any two arbitrary configurations. The same does not hold for pairs of Dubins or Reeds-Shepp vehicles, and a description of the reachable sets in these cases is provided. Finally, as direct extension of controllability results of pairs of identical vehicles, the controllability results for two kinds of formation of n identical vehicles are presented.**

I. Introduction

This paper provides the results of controllability for pairs of identical vehicles maintaining a constant distance. The controllability of a system answers the question about the existence of an admissible trajectory between any given two configurations, which is an important condition for a feasible design of motion planning ([1]) and for the existence of an optimal trajectory (see e.g. [2]). Moreover, the study of pairs of vehicles maintaining a constant distance helps the design of navigation strategies for a group of robots moving in formation (see e.g. [3], [4] and [5]).

In this paper we adopt the notation used in [2],[6]. A system is *controllable* if, for every pair of points p and q in the configuration space, there exists a control that steers the system from p to q. It is *small-time locally controllable* (STLC) from a point p if the set of points reachable before a given time T contains a neighborhood of p for any T. A control system will be said to be small-time controllable if it is small-time controllable from any point of the configuration space. The small-time controllability can be used to answer the problem about existence of collision-free admissible paths (see e.g. [1]). The challenging aspect in the controllability of the considered systems is that admissible controls depend on the configuration variables. Therefore, based on existing controllability theorems and on the accessibility rank condition of weakly reversible systems, we provide controllability theorems specific for such systems. Furthermore, conditions to verify the controllability of such systems are also provided.

This paper provides the results of controllability for the five most common types of robot vehicles which are widely discussed in the literature: Dubins [7], Reeds-Shepp (RS) [8], differential drive (DDV) [9], [10], car-like (Car) [11], [12] and convexified Reeds-Shepp (CRS) [2]. As a result, we show that while pairs of the latter three types of vehicle can be steered between any two arbitrary configurations, the same does not hold for pairs of vehicles of the first two types. For these two cases, a description of the reachable sets is provided. To the authors best knowledge, in the current literature no result on the controllability of pairs of vehicles that maintain a given distance is reported.

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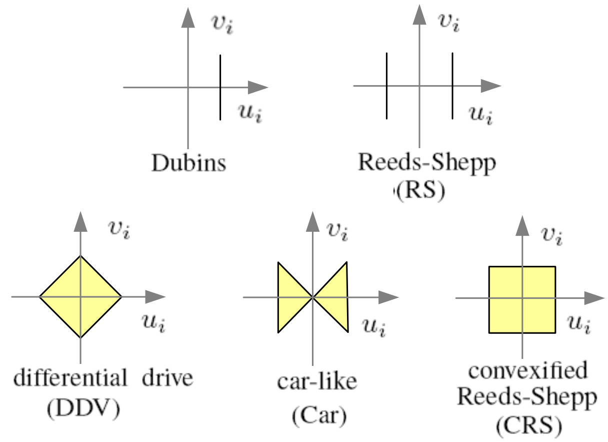

Fig. 1. The admissible controls for five types of robot vehicles.

II. Controllability Theorems

We first introduce the controllability theorems and lemmas that we will use in the following sections to prove controllability for the considered systems.

A. Controllability Definitions and Theorems

The systems we will study are affine control systems that can be written as

$$\Sigma_{\text{aff}} : \begin{cases} \dot{x} = f(x,u) = g_0(x) + \sum_{i=1}^{m} g_i u_i; \\ x \in \mathcal{X} \subseteq \mathbb{R}^n, u \in \mathcal{U}(x) \subseteq \mathbb{R}^m. \end{cases} \tag{1}$$

Let $\mathcal{A} := \{f_u = f(.,u), u \in \mathcal{U}\}$ be the set of system's vector fields.

Definition 1: The Lie algebra $\mathcal{A}_{LA}$ of vector fields $\mathcal{A}$ is called the *accessibility Lie algebra* associated to the system. The *accessibility rank condition (ARC)* holds at $x_0 \in \mathcal{X}$ if $\mathcal{A}_{LA}(x_0) = \mathbb{R}^n$.

Accessibility rank condition in [13] is also called controllability rank condition in [14], and Lie algebra rank condition in [2] and [15]. The verification of the accessibility rank condition is not straightforward. However, from [16] and [13] it holds

Lemma 1: If $0 \in \mathrm{conv}(\mathcal{U})$ and $\mathrm{aff}(\mathcal{U}) = \mathbb{R}^m$, then U is called almost proper and $\mathcal{A}_{LA} = \{g_0, \cdots, g_m\}_{LA}$.
Where $\mathrm{conv}(\mathcal{U})$ and $\mathrm{aff}(\mathcal{U})$ are the convex hull and the affine hull of $\mathcal{U}$, respectively.

Recall that a system is *symmetric* if every trajectory run backwards in time is also a trajectory.

Theorem 1: For a symmetric system, if the accessibility rank condition holds at every point $x_0 \in \mathcal{X}$, the system is STLC from every x_0. In particular, if $\mathcal{X}$ is connected, then it is controllable.
The theorem follows from results in [2], [16] and [13].

B. Proposed Controllability Theorems

From the results on complete controllability for a weakly reversible system which is stated in Theorem 2 in [13], we can state the controllability theorem for the special systems in which the control domains vary with configurations.

Definition 2: A system with state space $\mathcal{X}$ is *weakly reversible* if $x_1 \in \mathcal{R}(x_0)$ if and only if $x_0 \in \mathcal{R}(x_1)$, $\forall x_0, x_1 \in \mathcal{X}$.

Theorem 2: [13] For a weakly reversible system, if the accessibility rank condition holds at every state $x_0 \in \mathcal{X}$ and $\mathcal{X}$ is connected, then the system is completely controllable.
Notice that any symmetric system is definitely a weakly reversible system.

For systems in which there exists a set of points in $\mathcal{X}$ such that the ARC does not hold, controllability can still be ensured (based on a trivial extension of Theorem 2) whenever from this set it is possible to reach points in which the ARC holds:

Theorem 3: Given a weakly reversible affine control system, such as 1, with $\mathcal{X}$ connected, and given $S^1 \subset \mathcal{X}$ such that

1) $\forall x_0 \in S^1, U(x_0) \subseteq \mathbb{R}^m$ almost proper, $\mathcal{A}_{LA}(x_0) = \mathbb{R}^n$;
2) $\forall x_0 \in S^2 := \mathcal{X} \setminus S^1$, $U(x_0) \subseteq \mathbb{R}^l, l < m$, $\mathcal{A}_{LA}(x_0) \neq \mathbb{R}^n$; but $\mathcal{R}(x_0) \cap S^1 \neq \emptyset$,

then the system is completely controllable. Moreover, if it is symmetric, then it is also STLC.

Remark 1: Whenever S^2 does not have interior points and it is such that its boundary function $\Phi(x)$ is differentiable, a sufficient condition for $\mathcal{R}(x_0) \cap S^1 \neq \emptyset$ is that there exists an admissible control $\omega \in U(x_0)$, $x_0 \in S^2$ such that $\langle f(x_0, \omega), \frac{\partial \Phi}{\partial x} \rangle \neq 0$. This condition will be used to prove the controllability of Car and RS vehicles. And if $\langle f(x_0, \omega), \frac{\partial \Phi}{\partial x} \rangle = 0$, S^2 is invariant under all admissible control ω, hence the system is not controllable, see fig.2.

III. Kinematic Models

In this section the kinematic model for two identical vehicles (Dubins, Reeds-Shepp, differential drive, car-like and convexified Reeds-Shepp) traveling at constant distance is obtained starting from the kinematic model of a single vehicle. It is worthwhile noticing that the models will differ in the control set and not in the kinematics.

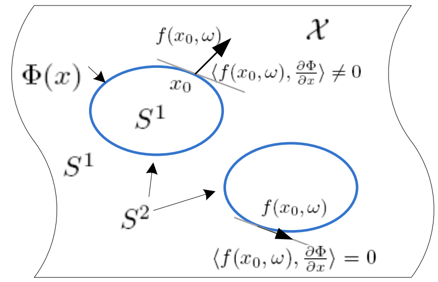

Fig. 2. Illustration of the sufficient condition for $\mathcal{R}(x_0) \cap S^1 \neq \emptyset$ when $int(S^2) = \emptyset$ and its boundary function $\Phi(x)$ is differentiable.

A. Kinematic Models for Single Vehicles

The kinematic model of the considered vehicles can be described as

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \\ 0 \end{pmatrix} u_i + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_i \tag{2}$$

where $\xi_i = (x_i, y_i, \theta_i) \in \mathbb{R}^2 \times \mathcal{S}^1$ denotes a configuration of vehicle i, i.e. (x_i, y_i) is the position and θ_i is the forward direction angle with respect to the positive x-axis.

The controls u_i and v_i describe the linear and angular velocities of vehicle i, respectively. We write $(u_i, v_i) \in U$, where U is the admissible control domain. Fig. 1 shows the different admissible control domains for the above five types of vehicles. Without loss of generality, we consider normalized maximal and minimal velocities and assume that the minimum turning radius $R_{\min} = 1$ for Dubins, RS and car-like robots, although in order to emphasize its influence R_{min} often remains. For DDV the wheel angular velocities are bounded, hence the admissible control domain is a diamond (rhombus), i.e. $U_{\mathrm{DDV}} = \{(u_i, v_i))| 0 \le |u_i| \le 1; 0 \le |v_i| \le 1 - |u_i| \le 1\}$. For car-like vehicles, $U_{\mathrm{car}} = \{(u_i, v_i))| 0 \le |v_i| \le |u_i| \le 1\}$. A Dubins vehicle is a car-like vehicle which is only able to move forward with constant velocity, i.e. $U_{\mathrm{Dubins}} = 1 \times [-1, 1]$. RS vehicles, can move both forward and backward at constant velocity 1, i.e. $U_{\mathrm{RS}} = \{-1, 1\} \times [-1, 1]$. For CRS robots, $U_{\mathrm{CRS}} = [-1, 1] \times [-1, 1]$ is obtained by convexifying U_{RS} and CRS is the kinematic model of a tricycle.

B. Kinematic Models for Pairs of Vehicles

Consider a pair of vehicles (ξ_1 and ξ_2) traveling while maintaining a constant distance D. Let ϕ denote the angle of vector $(x_2 - x_1, y_2 - y_1)$ with respect to the x-axis, see fig.3. Thus we can write:

$$x_2 - x_1 = D\cos\phi; \; y_2 - y_1 = D\sin\phi. \tag{3}$$

We choose $\xi_{1-2} = (x_1, y_1, \theta_1, \phi, \theta_2)$ as the configuration vector of the system consisting of two identical vehicles maintaining a constant distance. The nonholonomic constraint for each vehicle is:

$$\dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i = 0.$$

From (3), we have that $\dot{x}_2 = -D\dot{\phi}\sin\phi + \dot{x}_1$ and $\dot{y}_2 = D\dot{\phi}\cos\phi + \dot{y}_1$.

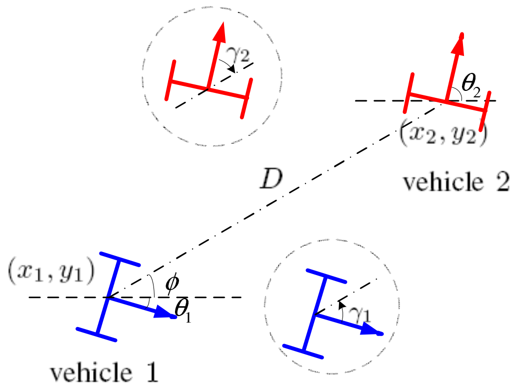

Fig. 3. The kinematic model of pairs of identical vehicles.

Finally, the constraints for a pair of vehicles maintaining distance D can be written as:

$$\begin{aligned} \dot{x}_1 \sin\theta_1 - \dot{y}_1 \cos\theta_1 = 0 \\ (-D\dot{\phi}\sin\phi + \dot{x}_1)\sin\theta_2 - (D\dot{\phi}\cos\phi + \dot{y}_1)\cos\theta_2 = 0. \end{aligned} \tag{4}$$

Hence, a system of 5 unknowns ($\dot{x}_1$, $\dot{y}_1$, $\dot{\theta}_1$, $\dot{\phi}$, $\dot{\theta}_2$) and 2 linear equations (4) have been obtained. By computing the null space of the constraint matrix we obtain that there exists u such that the kinematic model of a pair of identical vehicles maintaining distance D is:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\phi} \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} D\cos\theta_1\cos(\phi-\theta_2) \\ D\sin\theta_1\cos(\phi-\theta_2) \\ 0 \\ \sin(\theta_2-\theta_1) \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v_2 \tag{5}$$

Let $q = (x_1, y_1, \theta_1, \gamma_1, \gamma_2)$ be the new configuration of the system with $\gamma_1 = \phi - \theta_1,\ \gamma_2 = \phi - \theta_2$. The kinematic model of a pair of identical vehicles maintaining distance D can be written as:

$$\dot{q} = f_1 u + f_2 u_1 + f_3 u_2, \tag{6}$$

where the system vector fields are:

$$f_1 = \begin{pmatrix} D\cos\theta_1\cos\gamma_2 \\ D\sin\theta_1\cos\gamma_2 \\ 0 \\ \sin(\gamma_1-\gamma_2) \\ \sin(\gamma_1-\gamma_2) \end{pmatrix}; f_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}; \ f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{7}$$

The relationship between u in (6) and u_1 and u_2 must be found. To maintain distance D, the velocity of both vehicles along distance direction should be the same, i.e.

$$u_1\cos\gamma_1 = u_2\cos\gamma_2. \tag{8}$$

Hence, from (2) and (6), we obtain that

$$u_1 = uD\cos\gamma_2; u_2 = uD\cos\gamma_1. \tag{9}$$

The systems of pairs of identical vehicles maintaining a constant distance will be denoted by Σ_{DDV}, Σ_{Car}, Σ_{Dubins}, Σ_{RS} and Σ_{CRS} for differential drive, car-like, Dubins, RS and CRS vehicles, respectively.

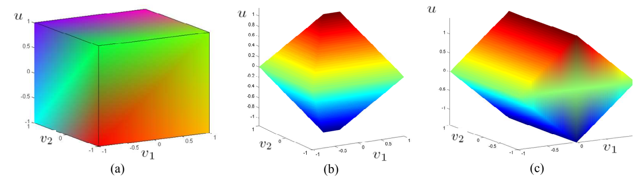

Fig. 4. The admissible $\mathcal{U}_{DDV}$ at configurations with (a): $\gamma_1 = \gamma_2 = \frac{\pi}{2}$; (b): $\gamma_1 = \frac{\pi}{6}, \gamma_2 = \frac{\pi}{4}$; and (c): $\gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{4}$.

IV. Controllability for DDV, Car and CRS systems

A. Controllability for DDV

Theorem 4: Σ_{DDV} is STLC and controllable on the configuration space $\mathcal{M}_{DDV} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \mathcal{S}^1$.

Proof: For DDV, the control is $|u_i| \leq 1$, and no constraint limits the configuration variables θ_1, γ_1 and γ_2. Hence the configuration space is $\mathcal{M}_{DDV} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \mathcal{S}^1$

Moreover, from (8) and $|u_i| \leq 1$ it follows

$$|u_1| \leq \min\left\{\frac{|\cos\gamma_2|}{|\cos\gamma_1|}, 1\right\}; \ |u_2| \leq \min\left\{\frac{|\cos\gamma_1|}{|\cos\gamma_2|}, 1\right\}. \tag{10}$$

From $|v_i| \leq 1 - |u_i| \leq 1$, we obtain that if $\gamma_1 = \gamma_2 = \frac{\pi}{2}$, its admissible control set is $\mathcal{U}_{DDV} = \{(v_1, v_2, u) | |v_i| \leq 1; |u| \leq 1\}$, shown in fig. 4 (a); otherwise, $\mathcal{U}_{DDV} = \left\{(v_1, v_2, u) | |v_i| \leq 1; |u| \leq \min\left\{\frac{1-|v_1|}{\cos\gamma_2}, \frac{1-|v_1|}{\cos\gamma_1}\right\}\right\}$, shown in fig.4 for two kinds of configuration with (b): $\gamma_1 = \frac{\pi}{6}, \gamma_2 = \frac{\pi}{4}$; (c):$\gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{4}$. Thus for any configuration, it satisfies $0 \in \text{conv}(\mathcal{U})$ and $\text{aff}(\mathcal{U}) = \mathbb{R}^m$.

Computing the vector fields (7), we have $f_4 = [f_1, f_2]$= $(D\sin\theta_1\cos\gamma_2,\ -D\cos\theta_1\cos\gamma_2,\ 0,\ \cos(\gamma_1-\gamma_2),\ \cos(\gamma_1-\gamma_2))^T$ and $f_5 = [f_1, f_3]$= $(-D\cos\theta_1\sin\gamma_2,\ -D\sin\theta_1\sin\gamma_2,\ 0,\ -\cos(\gamma_1-\gamma_2),\ -\cos(\gamma_1-\gamma_2))^T$. Notice that $\text{rank}([f_1, \dots, f_5]) = 5$, hence ARC holds at every $q \in \mathcal{M}_{DDV}$. Σ_{DDV} is also symmetric, thus from Theorem 1, Σ_{DDV} is STLC. Moreover $\mathcal{M}_{DDV}$ is connected, and the system is controllable. ∎

B. Controllability for CRS

Theorem 5: Σ_{CRS} is STLC and controllable on the configuration space $\mathcal{M}_{CRS} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \mathcal{S}^1$.

Proof: $\mathcal{M}_{CRS} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \mathcal{S}^1$ follows directly from $|u_i| \leq 1$ with the same reasoning used in the previous theorem.

Similarly to the proof of Theorem 4, the admissible control set is $\mathcal{U}_{CRS} = \{(v_1, v_2, u) | |v_i| \leq 1; |u| \leq 1\}$ if $\gamma_1 = \gamma_2 = \frac{\pi}{2}$, shown in fig. 5 (a); otherwise, $\mathcal{U}_{CRS} = \left\{(v_1, v_2, u) | |v_i| \leq 1; |u| \leq \min\left\{\frac{1}{D|\cos\gamma_2|}, \frac{1}{D|\cos\gamma_1)|}\right\}\right\}$, shown in fig.5 (b) at a specified configuration. The admissible control set $\mathcal{U}_{CRS}$ is proper for all configurations, Σ_{CRS} is symmetric and $\mathcal{M}_{CRS}$ is connected. Hence the thesis. ∎

C. Controllability for Car

Theorem 6: Σ_{car} is STLC and controllable on the configuration space $\mathcal{M}_{car} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \mathcal{S}^1$.

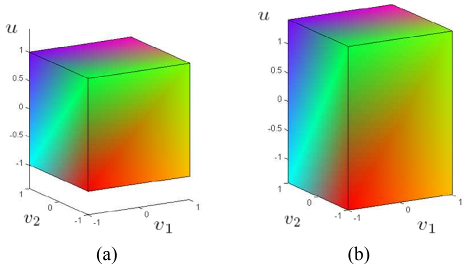

Fig. 5. The admissible $\mathcal{U}_{CRS}$ at configurations with (a): $\gamma_1 = \gamma_2 = \frac{\pi}{2}$; (b): $\gamma_1 = \frac{\pi}{6}, \gamma_2 = \frac{\pi}{4}$.

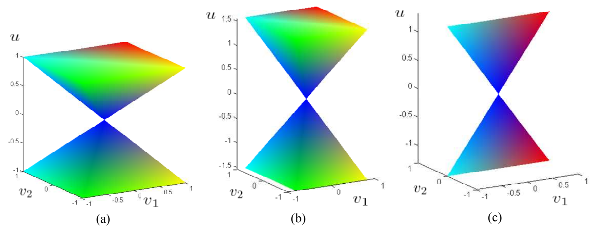

Fig. 6. The admissible $\mathcal{U}_{car}$ at configurations with (a): $\gamma_1 = \gamma_2 = \frac{\pi}{2}$; (b):$\gamma_1 = \frac{\pi}{6}, \gamma_2 = \frac{\pi}{4}$; and (c): $\gamma_1 = \frac{\pi}{2}, \gamma_2 = \frac{\pi}{4}$.

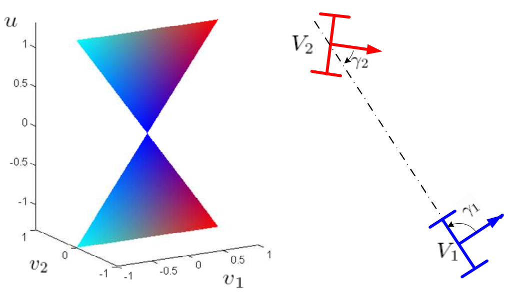

Fig. 7. The admissible $\mathcal{U}_{car}$ at a configuration with $\cos\gamma_1 = 0$ and $\cos\gamma_2 \neq 0$.

Proof: $\mathcal{M}_{car} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \mathcal{S}^1$ follows directly from $|u_i| \leq 1$. From (10) and $|v_i| \leq |u_i| \leq 1$, we can get that if $\gamma_1 = \gamma_2 = \frac{\pi}{2}$, $\mathcal{U}_{Car} = \{(v_1, v_2, u) | |v_i| \leq 1; |u| \leq 1\}$; otherwise $\mathcal{U}_{Car} = \left\{(v_1, v_2, u) | |v_1| \leq \min\left\{\frac{|\cos\gamma_2|}{|\cos\gamma_1|}, 1\right\}; |v_2| \leq \min\left\{\frac{|\cos\gamma_1|}{|\cos\gamma_2|}, 1\right\}; |u| \leq \min\left\{\frac{1}{D|\cos\gamma_2|}, \frac{1}{D|\cos\gamma_1|}\right\}\right\}$. As shown in fig.6, $\mathcal{U}_{Car}$ is almost proper at all configurations except at $\gamma_i = \frac{\pi}{2}$ and $\gamma_j \neq \frac{\pi}{2}, i, j = 1, 2$. For such configurations U is shown in fig. 6(c) and $\mathrm{aff}(\mathcal{U}_{Car}) = \mathbb{R}^2$. Considering $\gamma_1 = \frac{\pi}{2}, \gamma_2 \neq \frac{\pi}{2}$ (see fig.7), we have $\Phi(x) = \gamma_1 - \frac{\pi}{2}$. If we choose $u_1 = 1, v_1 = -1$, then $f(x_0, \omega) = (*, *, *, 1/D + 1, *)$ and $\frac{\partial\Phi}{\partial x} = (0, 0, 0, 1, 0)$. Thus $\langle f(x_0, \omega), \frac{\partial\Phi}{\partial x}\rangle > 0$. Thus the thesis follows from Theorem 3. ∎

V. Controllability for RS

For RS vehicles, $u_i = \pm 1$. Hence, from (8), we have:

$$\cos\gamma_1 = \pm\cos\gamma_2. \tag{11}$$

Four possible angular relationships between two vehicles can thus be obtained (see fig. 8):

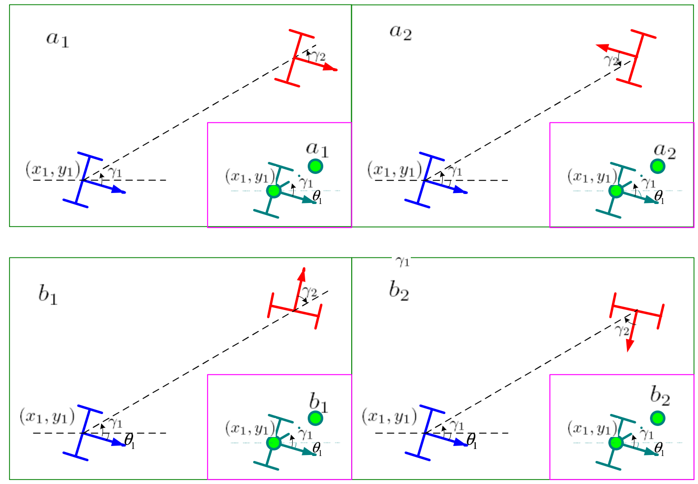

Fig. 8. Four angular relationships for Σ_{RS} and configuration representation by 4-dimensional parameters plus angular relationships $\{a_1, a_2, b_1, b_2\}$.

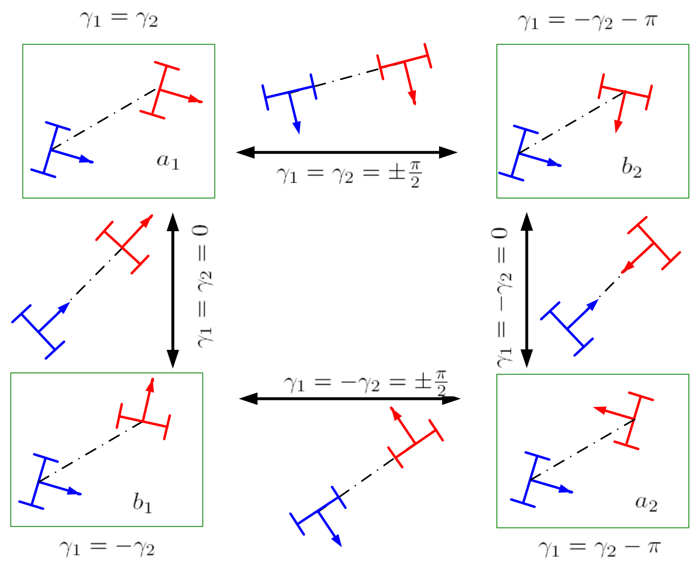

Fig. 9. Four angular relationships for pairs of RS vehicles and the combined cases.

$$\begin{array}{ll} a_1 : \gamma_1 = \gamma_2; & a_2 : \gamma_1 = \gamma_2 - \pi; \\ b_1 : \gamma_1 = -\gamma_2; & b_2 : \gamma_1 = -\gamma_2 - \pi. \end{array} \tag{12}$$

For simplicity and clarity of configurations representation for Σ_{RS}, we reduce the variables to 4 (x_1, y_1, θ_1, γ_1) and we use a parameter (a_1, a_2, b_1, or b_2) to denote the angular relationship (12). In fig.9 four possible angular relationships are represented together with the shared cases: $|\gamma_1| = |\gamma_2| = 0$ and $|\gamma_1| = |\gamma_2| = \frac{\pi}{2}$.

We denote with Σ_{RS}^A the system Σ_{RS} when relation a_1 or a_2 holds (in this case $v_1 = v_2$). From (5), the kinematic model of Σ_{RS}^A is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} v_1, \tag{13}$$

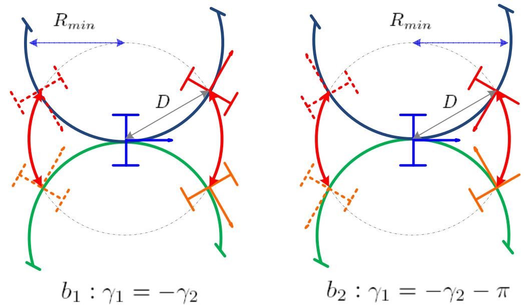

Fig. 10. Feasible configurations Σ_{RS}^B when $D = R_{\min}$.

with $u_1 \in \{-1, 1\}$ and $v_1 \in [-1, 1]$.

Remark 2: Notice that $\dot{\gamma}_1 = -\dot{\theta}_1$, hence there always exists a control (u_1, v_1) that steers γ_1 between any two values keeping $\phi = \gamma_1 + \theta_1$ constant.

We denote with Σ_{RS}^B the system Σ_{RS} when relation b_1 or b_2 holds (in this case $v_2 = \frac{4 \sin \gamma_1}{D} u_1 - v_1$). From $uD \cos \gamma_2 = u_1$ and (11), we have that the kinematic model of Σ_{RS}^B is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \\ 0 \\ \frac{2 \sin \gamma_1}{D} \\ \frac{-2 \sin \gamma_1}{D} \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} v_1. \tag{14}$$

Let the 4-dimensional system $\tilde{\Sigma}_{RS}^B$ be system Σ_{RS}^B projected on the first four coordinates. Hence, the configuration of $\tilde{\Sigma}_{RS}^B$ is $\tilde{q} = (x_1, y_1, \theta_1, \gamma_1)$ and the vector fields are

$$g_1 = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \\ 0 \\ \frac{2 \sin \gamma_1}{D} \end{pmatrix}; \; g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \tag{15}$$

Therefore, the kinematic model of $\tilde{\Sigma}_{RS}^B$ is:

$$\dot{\tilde{q}} = g_1 u_1 + g_2 v_1, \tag{16}$$

where $u_1 \in \{-1, 1\}$ and

$$\max \left\{ -1, \frac{4 u_1 \sin \gamma_1}{D} - 1 \right\} \leq v_1 \leq \min \left\{ \frac{4 u_1 \sin \gamma_1}{D} + 1, 1 \right\}. \tag{17}$$

To satisfy (17) we have $\frac{4 u_1 \sin \gamma_1}{D} - 1 \leq 1$ and $1 + \frac{4 u_1 \sin \gamma_1}{D} \geq -1$, hence

$$|\sin \gamma_1| \leq \frac{D}{2}. \tag{18}$$

It is now important to explicit the dependence of the results with respect to $R_{\min}$. Indeed, (18) would be $|\sin \gamma_1| \leq \frac{D}{2R_{\min}}$. If $D > 2R_{\min}$, $\gamma_1 \in \mathcal{S}^1$. On the other hand, if $D \leq 2R_{\min}$ $\gamma_1 \in \Gamma_I \cup \Gamma_{II} \cup \Gamma_s$, where

$$\begin{array}{l} \Gamma_I =]-\arcsin(\frac{D}{2}), \arcsin(\frac{D}{2})[, \\ \Gamma_{II} =]-\arcsin(\frac{D}{2}) + \pi, \arcsin(\frac{D}{2}) + \pi[, \\ \Gamma_s = \{\gamma_1 | |\sin \gamma_1| = \frac{D}{2}\}. \end{array} \tag{19}$$

For example, for $D = R_{\min}$ feasible configurations are represented in fig. 10 and admissible controls ($\Gamma_I =]-\frac{\pi}{6}, \frac{\pi}{6}[$, $\Gamma_{II} =]\frac{5\pi}{6}, \frac{7\pi}{6}[$, and $\Gamma_s = \{-\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}\}$) are represented in fig. 11.

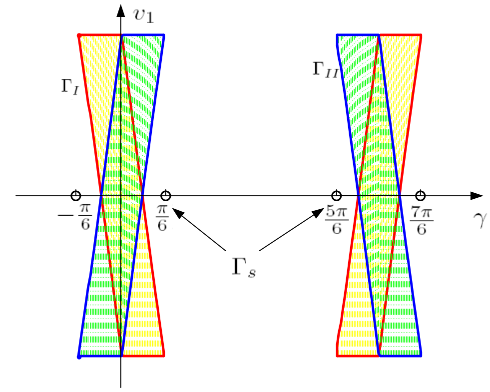

Fig. 11. If $D = R_{\min}$, the admissible control v_1 with respect to γ_1.

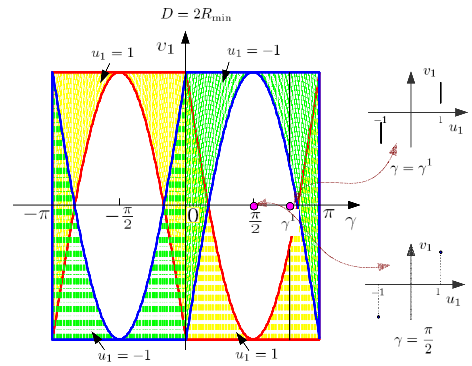

Fig. 12. If $D = 2R_{\min}$, the admissible control v_1 with respect to γ_1.

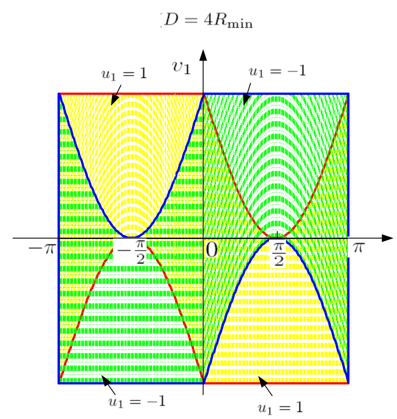

Fig. 13. If $D = 4R_{\min}$, the admissible control v_1 with respect to γ_1.

We denote with $\tilde{\mathcal{M}}_{RS}^{B+} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1$ the configuration space when $D > 2R_{\min}$, and with $\tilde{\mathcal{M}}_{RS}^{B-} = \mathbb{R}^2 \times \mathcal{S}^1 \times \Gamma_I \cup \Gamma_{II}$ and $\tilde{\mathcal{M}}_{RS}^{B^s} = \mathbb{R}^2 \times \mathcal{S}^1 \times \Gamma_s$ the configuration space when $D \leq 2R_{\min}$. Notice that $\tilde{\mathcal{M}}_{RS}^{B^s}$ consists of singular configurations.

Lemma 2: For $\tilde{\Sigma}_{RS}^B$, ARC holds at any $\tilde{q} \in \tilde{\mathcal{M}}_{RS}^{B+}$ if $D > 2R_{\min}$ and $\tilde{q} \in \tilde{\mathcal{M}}_{RS}^{B-}$ if $D \leq 2R_{\min}$. But ARC fails at $\tilde{q} \in \tilde{\mathcal{M}}_{RS}^{B^s}$.

Proof: We start applying remark 1 for $D \leq 2R_{\min}$ and $\tilde{q} \in \tilde{\mathcal{M}}_{RS}^{B^s}$. In this case $\sin \gamma_1 = \frac{D}{2R_{\min}}$. The only two admissible controls are either $(u_1^1, v_1^1) = -(u_1^2, v_1^2) = \pm(1, 1)$ or $(u_1^1, v_1^1) = -(u_1^2, v_1^2) = \pm(-1, 1)$, see fig.12. Notice that

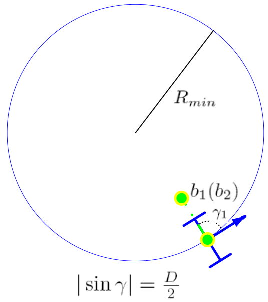

Fig. 14. All $\tilde{q} \in \tilde{\mathcal{M}}^{B^s}_{RS}$ can only reached points on a circle through $\tilde{q}$ of radius R_{min}.

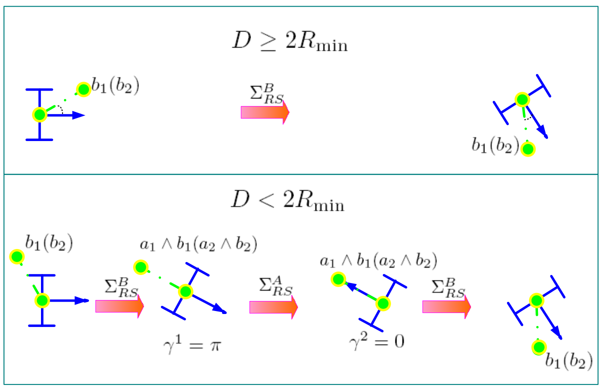

Fig. 15. Σ^B_{RS} is controllable for any given distance D.

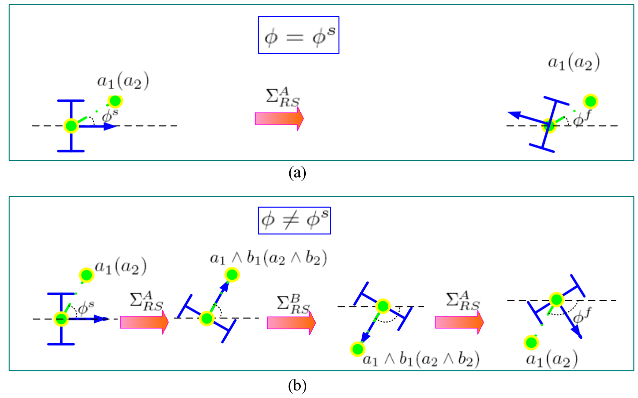

Fig. 16. Σ^A_{RS} is controllable for both $\phi = \phi^s$ and $\phi \neq \phi^s$.

$\mathrm{aff}(\mathcal{U}) \neq \mathbb{R}^2$ implies that only one motion direction is feasible at $\tilde{q}$. In this case $\Phi(x) = \gamma_1 - \arcsin(\frac{D}{2R_{min}})$. For all possible controls, $f(x_0, \omega) = (*, *, *, 0)$ and $\frac{\partial \Phi}{\partial x} = (0, 0, 0, 1)$. Thus $\langle f(x_0, \omega), \frac{\partial \Phi}{\partial x} \rangle = 0$. Hence, all reachable configurations from $\tilde{q}^0 = (x^0_1, y^0_1, \theta^0_1, \gamma^0_1) \in \tilde{\mathcal{M}}^{B^s}_{RS}$ lays on a circle with radius $R_{\min}$ and centered at $(x^0_1 - R_{\min} \sin\theta^0_1 \mathrm{sign}(\sin\gamma^0_1), y^0_1 + R_{\min}\cos\theta^0_1 \mathrm{sign}(\sin\gamma^0_1))$, see fig.14.

For $\tilde{q} \in \tilde{\mathcal{M}}^{B^+}_{RS}$, or $\tilde{q} \in \tilde{\mathcal{M}}^{B^-}_{RS}$, $0 \in \mathrm{conv}(\mathcal{U})$ and $\mathrm{aff}(\mathcal{U}) = \mathbb{R}^m$, see fig.13, 11 and 12. From (15) and (16), we have:

$$[g_1, g_2] = \begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \\ 0 \\ \frac{2\cos\gamma_1}{D} \end{pmatrix}; [g_1, [g_1, g_2]] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{20}$$

The accessibility rank condition holds and hence the thesis. ∎

Lemma 3: $\tilde{\Sigma}^B_{RS}$ is STLC at any $\tilde{q} \in \tilde{\mathcal{M}}^{B^+}_{RS}$ and any $\tilde{q} \in \tilde{\mathcal{M}}^{B^-}_{RS}$.

Proof: $\tilde{\Sigma}^B_{RS}$ is symmetric because for any required distance D, at any $q \in \tilde{\mathcal{M}}^{B^+}_{RS}$ (or $\tilde{\mathcal{M}}^{B^-}_{RS}$), if (u_1, v_1) is a feasible control, then $(-u_1, -v_1)$ is also feasible, see fig. 12. From Theorem 1 and Lemma 2 the thesis follows. ∎

If $D \geq 2R_{\min}$, $\tilde{\mathcal{M}}^{B^+}_{RS}$ is connected, so we can get the following corollary.

Corollary 1: If $D \geq 2R_{\min}$, $\tilde{\Sigma}^B_{RS}$ is also controllable.

We are now able to prove controllability for the 5-dimensional system Σ_{RS}. With a slight abuse of notation we denote with $\mathcal{M}^A_{RS} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \{a_1, a_2\}$ and $\mathcal{M}^B_{RS} = \tilde{\mathcal{M}}^B_{RS} \times \{b_1, b_2\}$ the configuration spaces for Σ^A_{RS} and Σ^B_{RS}, respectively. Furthermore let, $\tilde{\mathcal{M}}^B_{RS} = \tilde{\mathcal{M}}^{B^+}_{RS}$ ($\tilde{\mathcal{M}}^{B^-}_{RS}$) if $D \geq 2R_{\min}$ ($D < 2R_{\min}$). Finally, let $\mathcal{M}^{A_i}_{RS}$ be associated to relations a_i and $\mathcal{M}^{B_i}_{RS}$ to relations b_i.

Theorem 7: Σ_{RS} is controllable on $\mathcal{M}_{RS} = \mathcal{M}^A_{RS} \cup \mathcal{M}^B_{RS}$.

Proof: Corollary 1 states that Σ^B_{RS} for $D \geq 2R_{\min}$ is controllable.

We now prove that for $D < 2R_{\min}$, the system can be steered between any two configurations in $\tilde{\mathcal{M}}^{B^-}_{RS}$ crossing $\mathcal{M}^A_{RS}$. Without loss of generality let $q^0 \in \tilde{\mathcal{M}}^B_{II} = \mathbb{R}^2 \times \mathcal{S}^1 \times \Gamma^B_{II}$, a trajectory from q^0 to $q^1 = (x_1, y_1, \theta_1, \pi)$ for some (x_1, y_1, θ_1) that evolves in $\tilde{\mathcal{M}}^B_{II}$ always exists for Lemma 3, see fig. 15. The system then evolves in $\mathcal{M}^A_{RS}$ (as Σ^A_{RS}), and can reach $q^2 = (\hat{x}_1, \hat{y}_1, \hat{\theta}_1, 0)$ for some $(\hat{x}_1, \hat{y}_1, \hat{\theta}_1)$ for Remark 2. Then system can evolves in $\tilde{\mathcal{M}}^B_I$ to achieve any $q^f \in \tilde{\mathcal{M}}^B_I$ for lemma 3. There exists an equivalent control law that steers the system from $q^0 \in \tilde{\mathcal{M}}^B_I$ to $q^f \in \tilde{\mathcal{M}}^B_{II}$.

We now prove that the system can be steered between any two configurations in $\tilde{\mathcal{M}}^A_{RS}$ crossing $\mathcal{M}^B_{RS}$. For $q^0 = (x^0_1, y^0_1, \theta^0_1, \gamma^0_1) \in \mathcal{M}^{A_i}_{RS}$, Remark 2 implies that any point with $\phi = \gamma_1 + \theta_1 = \gamma^0_1 + \theta^0_1 = \phi^0$ can be reached in $\mathcal{M}^{A_i}_{RS}$, see fig.16 (a). Referring to fig.16 (b), if the final point in $\mathcal{M}^{A_i}_{RS}$ is such that $\phi \neq \phi^0$ we proceed as follows: 1) from q^0, achieve a configuration $q^1 = (x^1_1, y^1_1, \theta^1_1, \gamma^1_1)$ with $\theta^1_1 = \phi^1 = \phi^0$, notice that $q^1 \in \mathcal{M}^B_{RS}$ with $\gamma^1_1 = 0$. 2) from q^1, reach q^2 with $\theta^2_1 = \phi^2 = \phi^f$, evolving with Σ^B. This is possible for the first part of this proof. 3) from q^2, reach q^f evolving according to Σ^A_{RS} (Remark 2).

Finally, the four systems $\Sigma^{A_i}_{RS}$ and $\Sigma^{B_i}_{RS}$ are controllable for each angular relationship. The switches between them are shown in fig.9, so that Σ_{RS} is controllable. ∎

VI. Controllability for Dubins

For Dubins vehicles, $u_i = 1$, hence from (8), we have:

$$\cos\gamma_1 = \cos\gamma_2. \tag{21}$$

Thus we have two possible angular relationships between two vehicles

$$a : \gamma_1 = \gamma_2;\ b : \gamma_1 = -\gamma_2. \tag{22}$$

Angular relationships and their intersection cases $a \wedge b : \gamma_1 = \gamma_2 = k\pi, k = 0, 1$ are reported in fig.17.

Using the same reasoning used for Σ_{RS}, when $a : \gamma_1 = \gamma_2$ ($v_1 = v_2$), the kinematic model of Σ^A_{Dubins} is

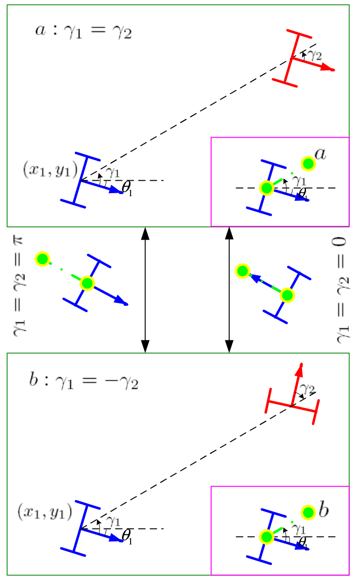

Fig. 17. Two angular relationships a, b for Σ_{Dubins}.

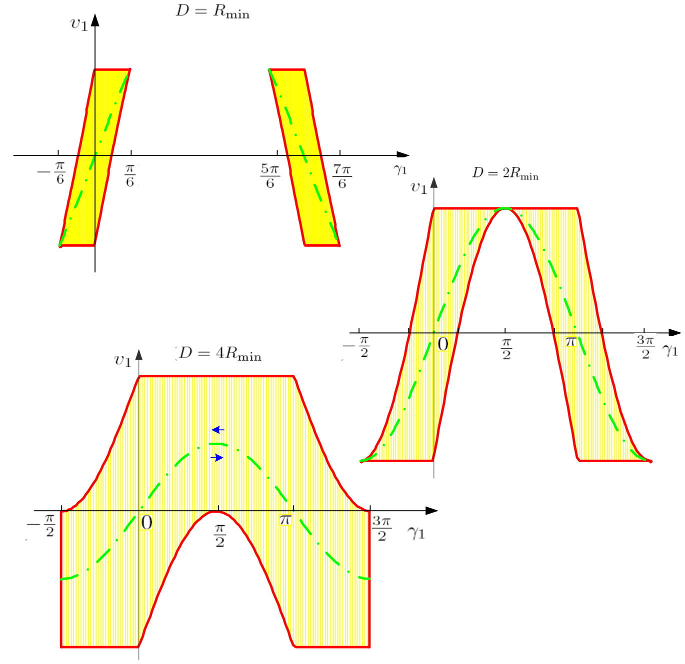

Fig. 18. The admissible controls $\mathcal{U}_{Dubins}$ with respect to γ_1.

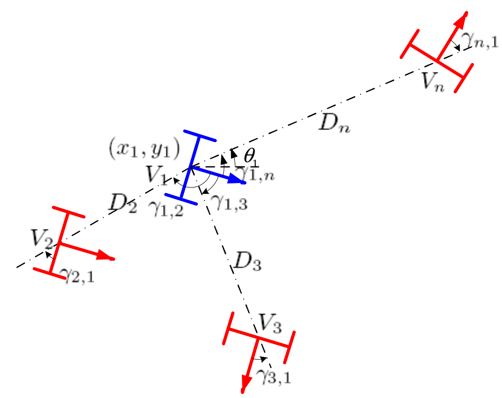

Fig. 19. The star formation for n vehicles.

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} v_1, \tag{23}
$$

where $u_1 \in \{-1, 1\}$ and $v_1 \in [-1, 1]$.

If two vehicles have the angular relationship b, then kinematics of Σ^B_{Dubins} is:

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \\ \frac{2\sin\gamma_1}{D} \\ \frac{-2\sin\gamma_1}{D} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} v_1. \tag{24}
$$

The range of v_1 is given by $\max\left\{-1, \frac{4\sin\gamma_1}{D} - 1\right\} \le v_1 \le \min\left\{\frac{4\sin\gamma_1}{D} + 1, 1\right\}$ and $|\sin\gamma_1| \le \frac{D}{2}$.

The controllability of Σ_{Dubins} requires similar reasoning as the controllability of Σ_{RS}. However, it is much more challenging to prove that Σ_{Dubins} is a weakly reversible system, details of the proof can be found in [17]. For space limitations, we only report the controllability results for Σ_{Dubins}.

Let $\mathcal{M}^A_{Dubins} = \mathbb{R}^2 \times \mathcal{S}^1 \times \mathcal{S}^1 \times \{a\}$ and $M^B_{Dubins} = \mathbb{R}^2 \times \mathcal{S}^1 \times \Gamma_{Dubins} \times b$. Let also $\Gamma_{Dubins} = \{\gamma_1 || \sin\gamma_1| < \frac{D}{2}\}$ and $\Gamma^s_{Dubins} = \{\gamma_1 || \sin\gamma_1| = \frac{D}{2}\}$.

Theorem 8: [17] Σ_{Dubins} is controllable on the configuration space $\mathcal{M}_{Dubins} = \mathcal{M}^A_{Dubins} \cup \mathcal{M}^B_{Dubins}$.

When $D \le R_{\min}$, if $q \in M^{B_s}_{Dubins} = \mathbb{R}^2 \times \mathcal{S}^1 \times \Gamma^s_{Dubins} \times b$, the reachable configurations is a limit circle.

VII. Controllability for n vehicles

This section gives a direct extension of above controllability results for pairs of vehicles to n identical vehicles both for a star formation with a leader and for a chain formation.

A. Controllability for n vehicles with star formation

Given n vehicles V_i, $i = 1, \cdots, n$, let V_1 be a leader. Assume the distances D_i, $i = 2, \cdots, n$ between V_1 to V_i are different such that no collision between vehicle occurs. Let $\gamma_{1,i}$ ($\gamma_{i,1}$), $i = 2, \cdots, n$ denote the angle from the heading direction of V_1 (V_i) to the distance direction from V_1 to V_i, see fig.19. Such system is denoted by Σ^n_s.

Let $\bar{q} = (x_1, y_1, \theta_1, \gamma_{1,2}, \gamma_{2,1}, \cdots, \gamma_{1,n}, \gamma_{n,1})$ be the configuration of Σ^n_s. If vehicles are all DDV, Car or CRS types, the configuration spaces can be written as $\bar{\mathcal{M}} = \mathbb{R}^2 \times \underbrace{\mathcal{S}^1 \times \cdots \times \mathcal{S}^1}_{2n-1}$. From Theorems4, 5 and 6 corresponding Σ^n_s are completely controllable.

For RS and Dubins vehicles, since for $D_i \le 2R_{min}$ the admissible control v_1 does not exist for all possible configurations, we assume that $D_i > 2R_{min}$ for all $i = 1, \ldots, n$. For RS vehicle, define $S^A = \mathcal{S}^1$ with angular relation $a_1 : \gamma_{1,i} = \gamma_{i,1}$ and $a_2 : \gamma_{1,i} = \gamma_{i,1} - \pi$; $S^B = \mathcal{S}^1$ with angular relation $b_1 : \gamma_{1,i} = -\gamma_{i,1}$ and $b_2 : \gamma_{1,i} = -\gamma_{i,1} - \pi$. For Dubins vehi-

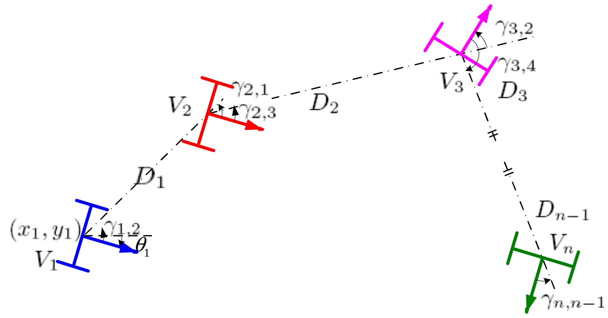

Fig. 20. The chain formation for n vehicles.

cles, define $S^A = \mathcal{S}^1$ with angular relation $a : \gamma_{1,i} = \gamma_{i,1}$ and $S^B = \mathcal{S}^1$ with angular relation $b : \gamma_{1,i} = -\gamma_{i,1}$. Then we can write $\bar{\mathcal{M}} = \mathbb{R}^2 \times \mathcal{S}^1 \times \underbrace{\mathcal{S}^1 \times S^A \cup S^B \times \cdots \times \mathcal{S}^1 \times S^A \cup S^B}_{n}$. From Theorems7 and 8 corresponding Σ_s^ns are completely controllable.

B. Controllability for n vehicles with chain formation

This part gives another extension of controllability results for chain formations consisting of n vehicles $V_i,\ i = 1, \cdots, n$. Assume the distances $D_i,\ i = 1, \cdots, n-1$ between V_i to V_{i+1} are specified such that no collision between vehicle occurs. Let $\gamma_{i,i+1}$ ($\gamma_{i+1,i}$), $i = 1, \cdots, n-1$ denote the angle from the heading direction of V_i (V_{i+1}) to the distance direction from V_i to V_{i+1}, see fig.20. Such system is denoted by Σ_c^n.

Let $\bar{q} = (x_1, y_1, \theta_1, \gamma_{1,2}, \gamma_{2,1}, \cdots, \gamma_{n-1,n}, \gamma_{n,n-1})$ be the configuration of Σ_c^n. If vehicles are all DDV, Car or CRS types, the configuration spaces can be written as $\bar{\mathcal{M}} = \mathbb{R}^2 \times \underbrace{\mathcal{S}^1 \times \cdots \times \mathcal{S}^1}_{2n-1}$. From Theorems4, 5 and 6 corresponding Σ_c^n are completely controllable.

For RS and Dubins vehicles, we assume that all distance $D_i > 2R_{min}$. For RS vehicle, define $S^A = \mathcal{S}^1$ with angular relation $a_1 : \gamma_{i,i+1} = \gamma_{i+1,i}$ and $a_2 : \gamma_{i,i+1} = \gamma_{i+1,i} - \pi$; $S^B = \mathcal{S}^1$ with angular relation $b_1 : \gamma_{i,i+1} = -\gamma_{i+1,i}$ and $b_2 : \gamma_{i,i+1} = -\gamma_{i+1,i} - \pi$. For Dubins vehicles, define $S^A = \mathcal{S}^1$ with angular relation $a : \gamma_{i,i+1} = \gamma_{i+1,i}$ and $S^B = \mathcal{S}^1$ with angular relation $b : \gamma_{i,i+1} = -\gamma_{i+1,i}$. Then we can write $\bar{\mathcal{M}} = \mathbb{R}^2 \times \mathcal{S}^1 \times \underbrace{\mathcal{S}^1 \times S^A \cup S^B \times \cdots \times \mathcal{S}^1 \times S^A \cup S^B}_{n}$. From Theorems7 and 8 corresponding Σ_c^ns are completely controllable.

VIII. Conclusions

This paper has provided controllability results for pairs of identical vehicles (Dubins, Reeds-Shepp, differential drive, car-like and convexified Reeds-Shepp) that move maintaining a constant distance. Known theorems of controllability have been extended to solve the controllability problem for special affine control systems whose admissible control domains depend on their configurations. Furthermore, a practical condition has been provided to apply the proposed theorem for studied systems.

As a result, for differential drive, car-like and convexified Reeds-Shepp vehicles complete controllability has been proved. The same does not hold for pairs of Dubins or Reeds-Shepp vehicles, and a description of the reachable sets in these cases has been provided. Limit circles for particular configurations have been proved to exist in case of small distance to be maintained.

Finally, controllability results have been presented, as a direct extension of pairs, for n identical vehicles with star formations and chain formations. The optimal control for larger groups of robots are under study.

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